Entanglement Negativity and Quantum Magic in Fermionic Many Body Systems

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F.-H. Wang and X. Y. Xu, Nature Communications 16, 2637 (2025)
F.-H. Wang and X. Y. Xu, arXiv:2503.07731
J. Q. Fang and X. Y. Xu, arXiv:2503.07742
J. Q. Fang, F.-H. Wang, and X. Y. Xu, arXiv:2601.13314

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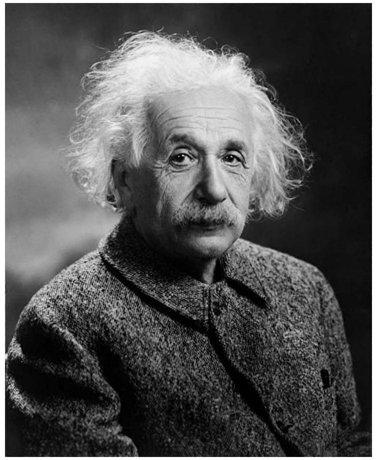
Quantum entanglement

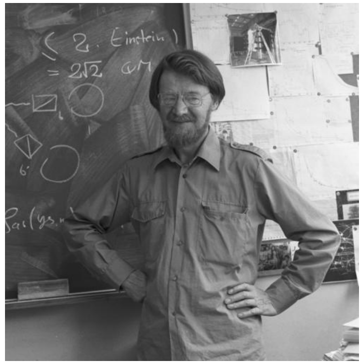
Einstein–Podolsky–Rosen paradox (1935)

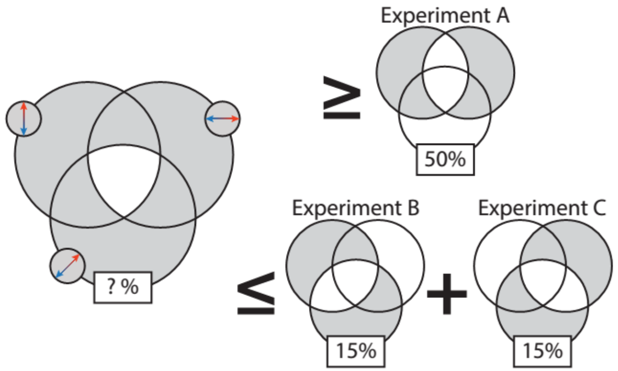

Albert Einstein

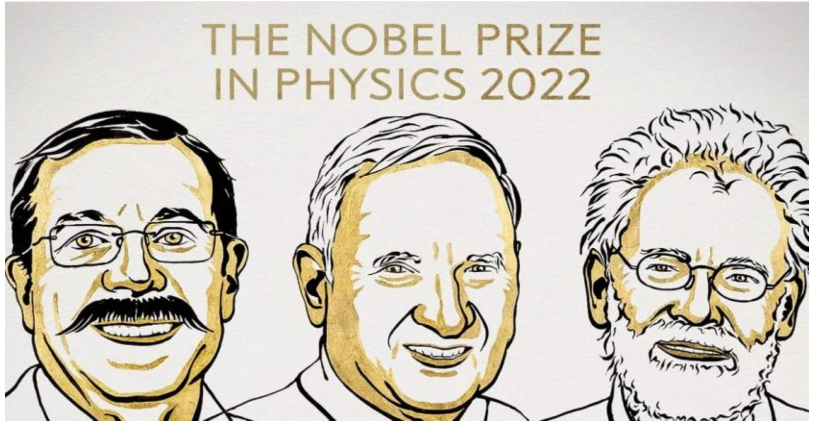

Boris Podolsky


Nathan Rosen

Bell's inequality (1964)


John Stewart Bell





The Nobel Prize in Physics 2022 was awarded jointly to **Alain Aspect**, **John F. Clauser** and **Anton Zeilinger** "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

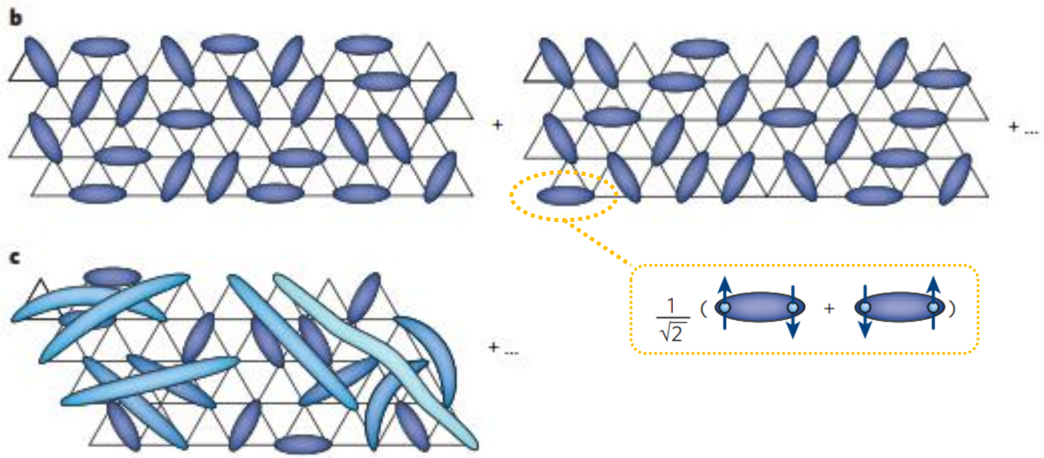
Violation of Bell inequalities is a demonstration of quantum entanglement!

From Bell state to quantum entangled matter

How many entanglement in Bell state?

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

How many entanglement in quantum matter?



Outline

Part I: Entanglement negativity

- Introduction: Mixed-state entanglement and negativity
- Fermionic partial transpose: From free fermions to interacting fermions
- Temperature-dependence of Rényi negativity in strongly-correlated fermionic models
- Twisted Rényi negativity vs untwisted Rényi negativity
- Stable computation of high-rank Rényi negativities
- Summary

Part II: From entanglement to magic

- Stabilizer Rényi entropy (SRE)
- two-point SRE for interacting fermion systems
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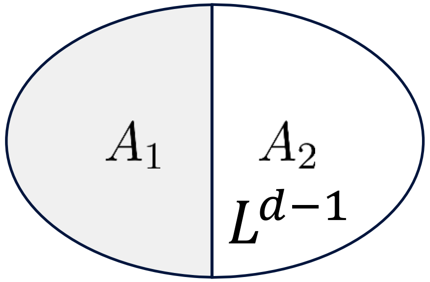
From pure-state to mixed-state entanglement

Rank-r (r-th) bipartite entanglement entropy

$$S_r(A_1) = \frac{1}{1-r}\ln\mathrm{Tr}_{A_1}\left[\left(\mathrm{Tr}_{A_2}\rho\right)^r\right] = a_0 L^{d-1}\ln L + a_1 L^{d-1} + a_2 L^{d-2}\ln L - \gamma$$

(mainly for 2D)



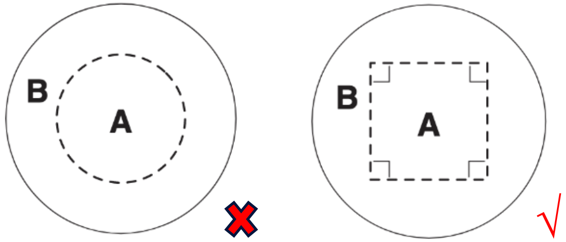
Beyond area-law behaviors for the classification of many-body ground states

Multiplicative Log

- 1D critical systems (1+1D CFT): $S_n \sim L^0 \ln L$
  - Holzhey, C., Larsen, F. & Wilczek, F. *Nuclear Physics B* **424**, 443–467 (1994)
  - Vidal, G., Latorre, J. I., Rico, E. & Kitaev, A. *Phys. Rev. Lett.* **90**, 227902 (2003)
  - Calabrese, P. & Cardy, J. *J. Stat. Mech.: Theor. Exp.* **2004**, P06002 (2004)
- d-dim Fermi surface: $S_n \sim L^{d-1}\ln L$
  - Gioev, D. & Klich, I. *Phys. Rev. Lett.* **96**, 100503 (2006)
  - Swingle, B. *Phys. Rev. Lett.* **105**, 050502 (2010)
  - Barthel, T., Chung, M.-C. & Schollwöck, U. *Phys. Rev. A* **74**, 022329 (2006)

Subleading Log

- Corners in 2D critical systems
  - Fradkin, E. & Moore, J. E. *Phys. Rev. Lett.* **97**, 050404 (2006)
  - Casini, H. & Huerta, M. *Nuclear Physics B* **764**, 183–201 (2007)

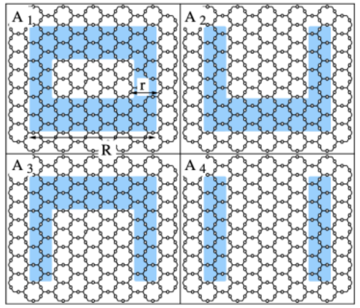


- Goldstone mode in symmetry-breaking phases
  - Metlitski, M. A. & Grover, T. arXiv:1112.5166 (2015)

Topological EE

- Non-local topological order
  - Kitaev, A. & Preskill, J. *Phys. Rev. Lett.* **96**, 110404 (2006)
  - Levin, M. & Wen, X.-G. *Phys. Rev. Lett.* **96**, 110405 (2006)
  - Jiang, H.-C., Wang, Z. & Balents, L. *Nature Phys* **8**, 902–905 (2012)

Non-local order

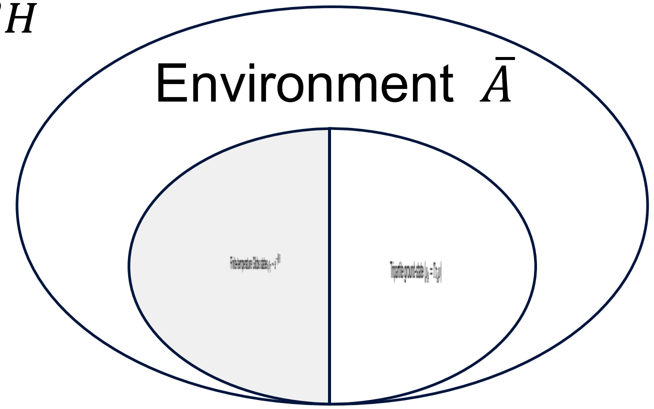


Levin-Wen scheme

From pure-state to mixed-state entanglement

Mixed states *exist in everyday life*

- Finite-temperature Gibbs states $\rho_T \sim e^{-\beta H}$ (with environment = a thermal bath)
- Tripartite ground-state ($\rho_A = \mathrm{Tr}_{\bar{A}}\rho$)
- Decoherence/measurement effects



Entanglement entropy contain both ***classical and quantum correlations****.*

Pure states

- Separable/product/unentangled states $|\Psi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle$
- Inseparable/entangled states: $|\Psi\rangle = \sum_i \sqrt{p_i}\left|\Phi_1^i\right\rangle \otimes \left|\Phi_2^i\right\rangle, \sum_i p_i = 1$ (Schmidt decomp.)

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Mixed states

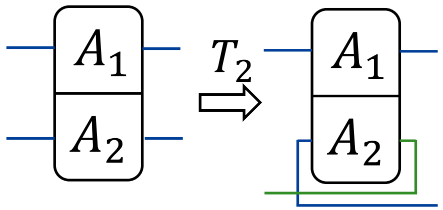
- Product/uncorrelated states: $\rho = \rho_A \otimes \rho_B$
- Separable states: $\rho = \sum_i p_i\, \rho_1^i \otimes \rho_2^i, p_i \geq 0, \sum p_k = 1$ (convex comb. of product states)
- Inseparable/entangled states

Consists of ***classical correlation***
But no ***quantum entanglement***

e.g., $\frac{1}{2}(|00\rangle\langle 00\rangle + |11\rangle\langle 11|) \Rightarrow S_r = \ln 2$

Logarithmic negativity (LN)

- Assert that a mixed state is separable——*separability* problem, is proved to be a NP-hard problem [1,2].
- Positive partial transpose (PPT) criteria [3,4]



  - Partial Transpose: $\left(|n_{A_1}, n_{A_2}\rangle\langle \bar{n}_{A_1}, \bar{n}_{A_2}|\right)^{T_2} = |n_{A_1}, \bar{n}_{A_2}\rangle\langle \bar{n}_{A_1}, n_{A_2}|$
  - For separable $\rho = \sum_i p_i \, \rho_1^i \otimes \rho_2^i$, $\rho^{T_2} = \sum_i p_i \, \rho_1^i \otimes \left(\rho_2^i\right)^T$ is still Hermitian, trace-1, and positive semi-definite——ρ^{T_2} has no negative eigenvalues.
- *Inseparability* detection: measures the degree of negative of ρ^{T_2} [5]

Negativity $\mathcal{N}(\rho_A) = \dfrac{||\rho^{T_2}||_1 - 1}{2}$

Logarithmic negativity $\mathcal{E}(\rho_A) = \ln||\rho^{T_2}||_1$

Trace norm

$$||\rho^{T_2}||_1 = \mathrm{Tr}\left[\sqrt{\rho^{T_2\dagger}\rho^{T_2}}\right]$$

  - $\mathcal{E} > 0 \Rightarrow$ entangled states
  - $\mathcal{E} = 0$ do not means ρ is separable in general (only for some simple low-dimensional cases [4,6])
  - **Entanglement monotone** (bosonic and fermionic)

[1] L. Gurvits, arXiv:quant-ph/0303055.
[2] S. Gharibian, QIC **10**, 343 (2010).
[3] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[4] M. Horodecki *et al.*, Physics Letters A **223**, 1 (1996).
[5] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[6] R. F. Werner and M. M. Wolf, Phys. Rev. Lett. **86**, 3658 (2001).

Rényi negativity

- Rényi negativity (r-moment of partially transposed density matrix (PTDM)):

$$\mathcal{E}_r = \frac{1}{1-r} \ln \mathrm{Tr}[(\rho^{T_2})^r]$$

  - analytically continued to the logarithmic negativity (LN): $\mathcal{E} = \lim_{r \to 1/2} (1-2r)\mathcal{E}_{2r}$
  - have no direct relation to entanglement or correlation

- Rényi negativity ratio (deviation from Rényi entropy):

$$R_r = \frac{1}{1-r} \ln \frac{\mathrm{Tr}\left[(\rho^{T_2})^r\right]}{\mathrm{Tr}[\rho^r]} = \mathcal{E}_r - S_r^{\mathrm{th}}$$

  - ***still not an entanglement monotone, but behave like LN in many cases.***

Rényi negativity ratio for Finite-T transitions

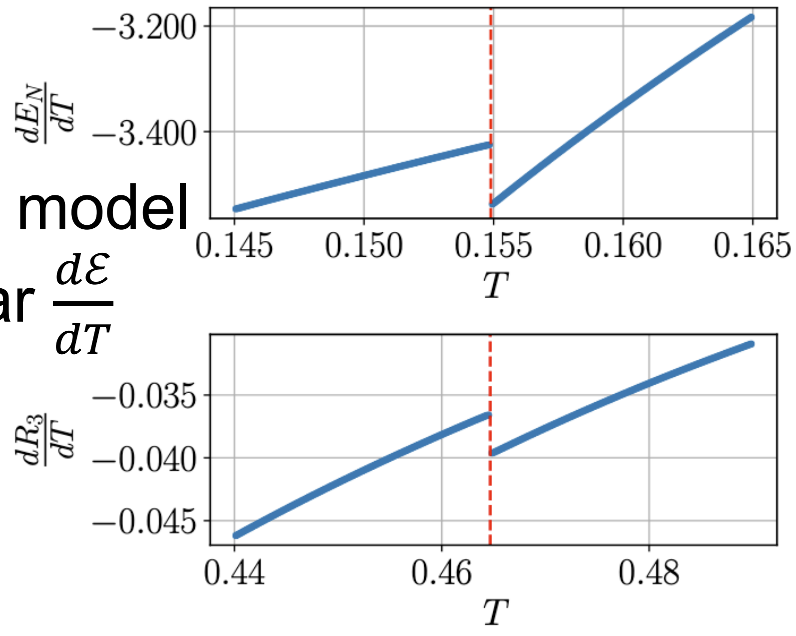
- rank-r RNR can be used to detect finite-temperature transitions
  similar behavior as LN, but the singular point is **at temperature rT_c**.

Ex.1: Quantum spherical model

$$H = \frac{1}{2} g \sum_{i=1}^{N} p_i^2 - \frac{1}{2N} \sum_{i,j=1}^{N} x_i x_j$$

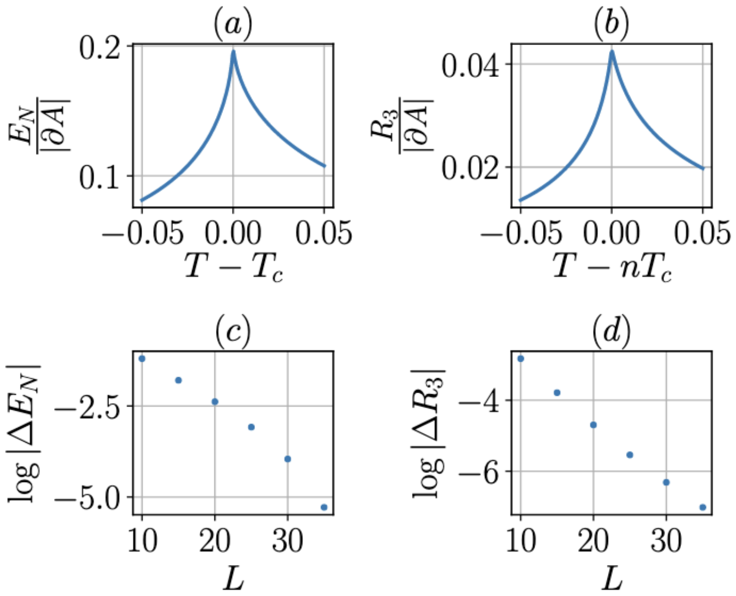
non-local model

$\Rightarrow$ singular $\frac{d\mathcal{E}}{dT}$



Ex.2: Mean-field description of 2D transversed field Ising model

$$H = \frac{1}{2} \sum_{\vec{r}} \left(\pi_{\vec{r}}^2 + m^2 \phi_{\vec{r}}^2\right) + \frac{1}{2} \sum_{\langle \vec{r}, \vec{r}' \rangle} K\left(\phi_{\vec{r}} - \phi_{\vec{r}'}\right)^2$$



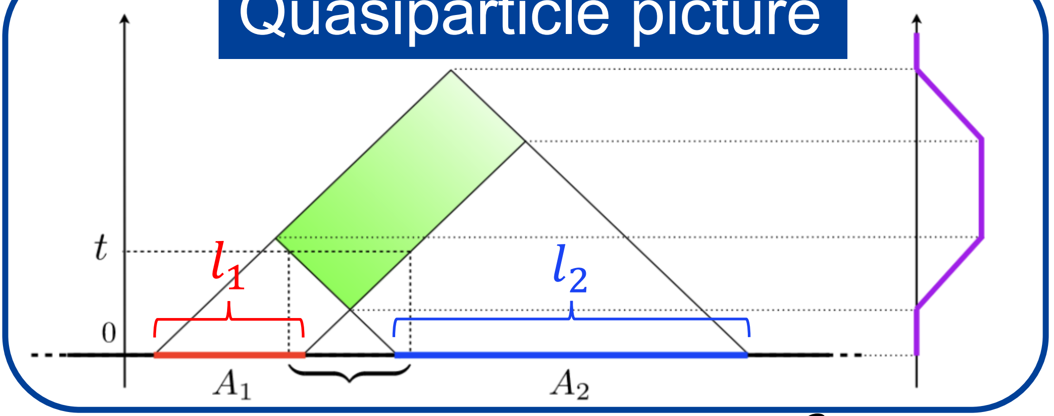
local model

- singular $\frac{d\left(\frac{\mathcal{E}}{\partial A}\right)}{dT}$
- exponential decay subleading terms

Rényi negativity ratio for quench dynamics

Ex.1: Quasiparticle picture for quenched integrable systems (harmonic chain)

Quasiparticle picture

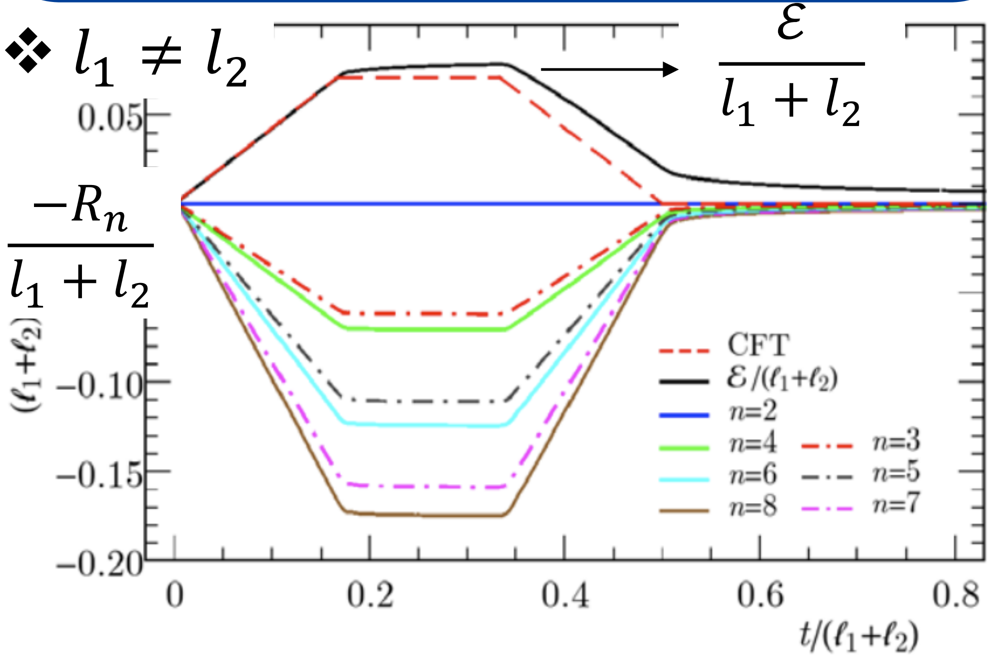


$$H(\omega) = \sum_{s=0}^{L-1} \left(\frac{1}{2m} p_s^2 + \frac{m\omega^2}{2} q_s^2 + \frac{K}{2} (q_{s+1} - q_s)^2 \right)$$

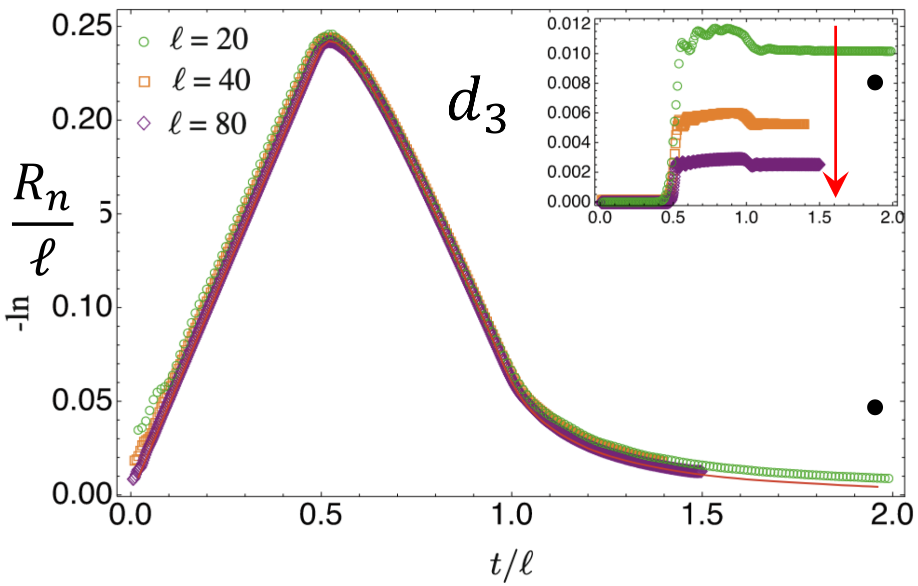
mass quench: $\omega = \omega_0 \to 0$

- similar behavior as LN *and* mutual information

❖ $l_1 \neq l_2$



❖ $l_1 = l_2 = \ell$

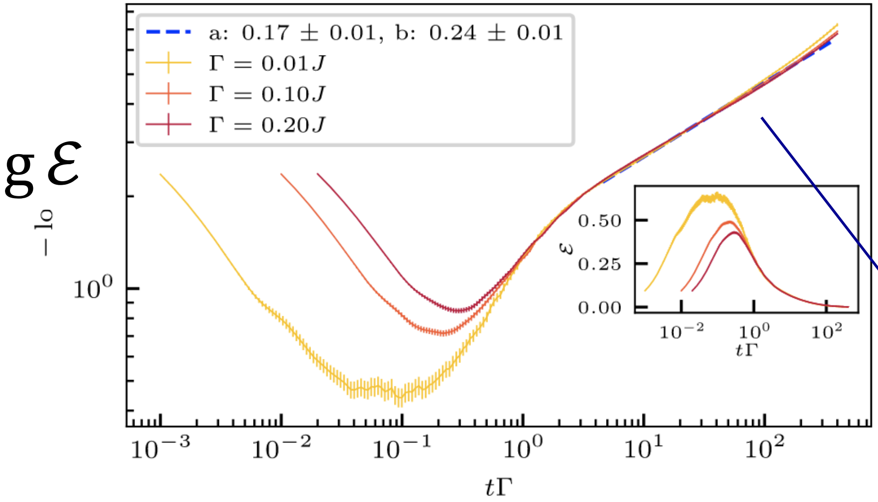


- $d_3 = R_3 - \frac{I^{(3)}_{A_1:A_2}}{2} \to 0$ in the space-time scaling limit (fixed t/ℓ, $\ell \to \infty$)
- For both bosonic and fermionic

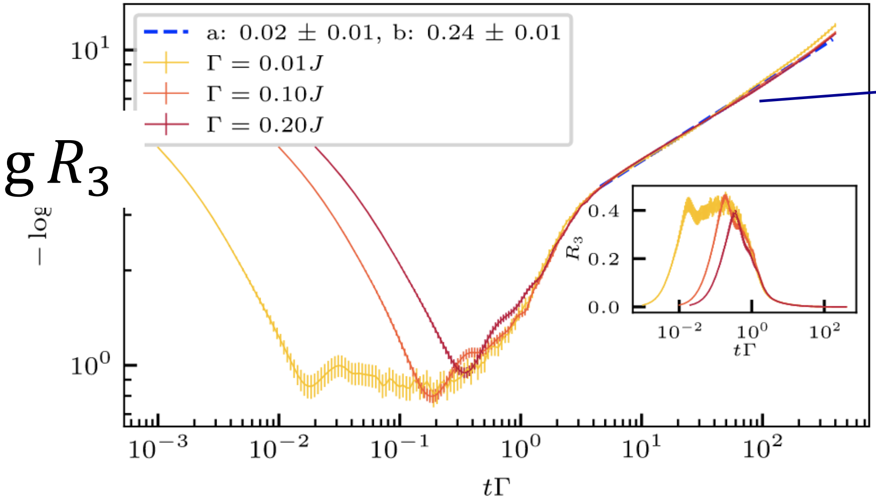
Rényi negativity ratio for quench dynamics

Ex.2: Random-field XXZ chain (quench + dephasing): probe many-body localization

$-\log \mathcal{E}$



$-\log R_3$



$$H = J\left[\sum_{i=1}^{L-1}\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right)\right] + \sum_{i=1}^{L} h_i S_i^z,$$

- ✓ ED $L = 8$
- ✓ Lindblad Eq. with decay rate Γ
- ✓ Quench from Neel state

$\mathcal{E}, R_3 \sim e^{\left(\frac{\Gamma t}{a}\right)^b}$ -fit with the same b

- captures the main negativity dynamic
- "*stays rather* ***blind*** *to the growing* ***classical correlations***"

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Fermionic partial transpose (FPT) in Fock space

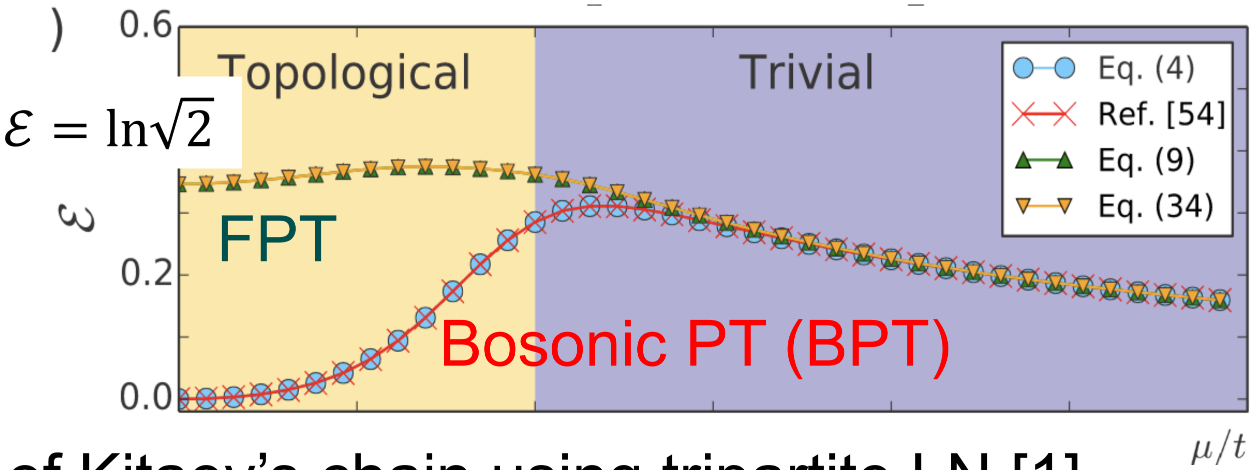
$$\left(|n_{A_1}, n_{A_2}\rangle\langle\bar{n}_{A_1}, \bar{n}_{A_2}|\right)^{T_2^f} = (-1)^{\phi(n,\bar{n})}\,|n_{A_1}, \bar{n}_{A_2}\rangle\langle\bar{n}_{A_1}, n_{A_2}|$$

$$(-1)^{\phi(n,\bar{n})} = \frac{[(\tau_2+\bar{\tau}_2) \bmod 2]}{2} + (\tau_1+\bar{\tau}_1)(\tau_2+\bar{\tau}_2)$$

n, n_{A_s}: (sub-)bit strings

$\tau_s = \sum_{j\in A_s} n_j$: norm of n_{A_s}

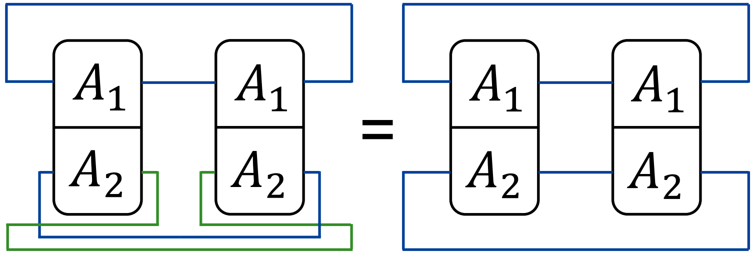
Properties	BPT	FPT [1]
Additivity $(\rho_1 \otimes \cdots \otimes \rho_n)^{T_2} = \rho_1^{T_2} \otimes \cdots \otimes \rho_n^{T_2}$ $\mathcal{E}(\rho_1 \otimes \cdots \otimes \rho_n) = \mathcal{E}(\rho_1) + \cdots + \mathcal{E}(\rho_n)$	×	√
$(\rho^{T_1})^{T_2} = \rho^T$	×	√



- ➢ Faithfully describe the topological phase of Kitaev's chain using tripartite LN [1]
- ➢ Fermionic LN is entanglement monotone, modified PPT criteria (even parity)
- ➢ Rank-2 RN is non-trivial [2]: $\mathrm{Tr}\left[\left(\rho^{T_2^f}\right)^2\right] = \mathrm{Tr}\left(\rho X_2 \rho X_2\right)$

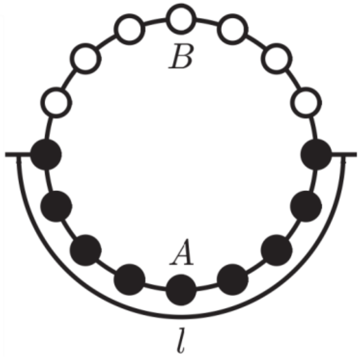
disorder operator: $X_2 = (-1)^{\sum_{j\in A_2} c_j^\dagger c_j}$

different from bosonic cases: $\mathrm{Tr}\left[\left(\rho^{T_2}\right)^2\right] = \mathrm{Tr}\left[\rho^2\right]$



Overlap matrix method for free fermions

A general free fermion system $$H = \sum_{i,j} h_{i,j} c_i^\dagger c_j = \sum_{\alpha=1}^{N} \epsilon_\alpha f_\alpha^\dagger f_\alpha$$



- Diagonal basis

$$f_\alpha^\dagger = \left(\boldsymbol{c}^\dagger U\right)_\alpha = \sum_{i=1}^{N} c_i^\dagger U_{i,\alpha}$$

Ground state $|\Psi\rangle = \prod_{\alpha=1}^{M} f_\alpha^\dagger |0\rangle$

- “Overlap matrix” basis

$$d_\gamma^\dagger = \sum_{\alpha=1}^{M} f_\alpha^\dagger \mathcal{U}_{\alpha,\gamma} = \sqrt{P_\gamma} d_{A_\gamma}^\dagger + \sqrt{1-P_\gamma} d_{B_\gamma}^\dagger$$

$$|\Psi\rangle = \det\left(\mathcal{U}^{-1}\right) \prod_{\gamma=1}^{M} d_\gamma^\dagger |0\rangle$$

$$\left\{d_{A_\gamma}, d_{A_{\gamma'}}^\dagger\right\} = \delta_{\gamma,\gamma'}$$

$$\left\{d_{B_\gamma}, d_{B_{\gamma'}}\right\} = \delta_{\gamma,\gamma'}$$

$$\left\{d_{A_\gamma}^{(\dagger)}, d_{B_{\gamma'}}^{(\dagger)}\right\} = 0$$

Overlap matrix

$$(M_A)_{\alpha\beta} = \sum_{i \in A} U_{i\alpha}^* U_{i\beta}$$

$$\mathcal{U}^\dagger M_A \mathcal{U} = \mathrm{diag}\left(\{P_\gamma\}_{\gamma=1}^{M}\right)$$

Density matrix

$$\rho = \bigotimes_{\gamma=1}^{M} \rho_\gamma$$

$\Longrightarrow$

Partial trace $\rho_A = \bigotimes_{\gamma=1}^{M} \rho_{A\gamma}$

Partial transpose $\rho^{T_B^f} = \bigotimes_{\gamma=1}^{M} \rho_\gamma^{T_B^f}$

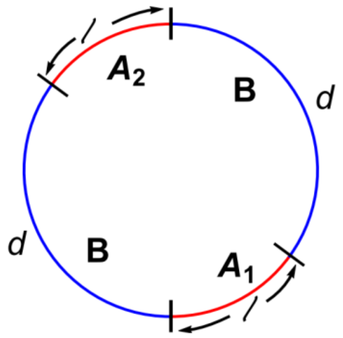
Overlap matrix method, tripartite case

overlap matrix basis

$$\rho_\gamma^{T_B^f}$$

occupation number basis

$$\tilde{\rho}_\gamma^{T_B^f}$$



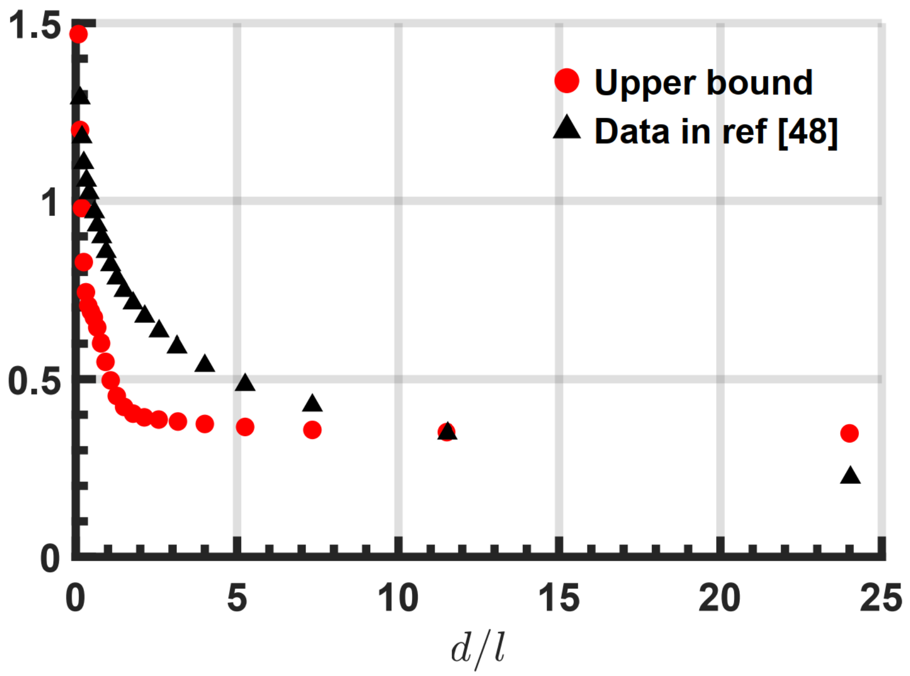
- For pure state, they are related with unitary transformation

$$\mathcal{E}\left(\rho_\gamma^{T_B^f}\right) = \mathcal{E}\left(\tilde{\rho}_\gamma^{T_B^f}\right)$$

- For mixed state, they are **NOT** related with unitary transformation

$$\mathcal{E}\left(\rho_{A\gamma}^{T_{A_1}^f}\right) \neq \mathcal{E}\left(\tilde{\rho}_{A\gamma}^{T_{A_1}^f}\right)$$

Breakdown in tripartite case



$$\mathcal{E}\left(\rho_A^{T_{A_1}^b}\right) \leq \mathcal{E}\left(\rho_A^{T_{A_1}^f}\right) + \ln\sqrt{2}$$

J. Eisert, V. Eisler, and Z. Zimboras, Phys. Rev. B 97, 165123 (2018)

Results in Chang et al. [1] violate the upper bound

Fermionic partial transpose of Gaussian states

- Gaussian states in Majorana basis: $\rho_0[\Gamma] = \frac{1}{Z}\exp(\frac{1}{4}\boldsymbol{\gamma}^T W \boldsymbol{\gamma}),\ \Gamma_{ij} \equiv \frac{\langle[\gamma_i,\gamma_j]\rangle}{2} = \tanh\left(-\frac{W}{2}\right)_{ij}$
- FPT in Majorana basis: $\rho_0^{T_2^f}[\Gamma] = \rho_0\left[\Gamma^{T_2^f}\right]$, with $\Gamma^{T_2^f} = \begin{pmatrix} \Gamma^{11} & \mathrm{i}\Gamma^{12} \\ \mathrm{i}\Gamma^{21} & -\Gamma^{22} \end{pmatrix}$ (**Wick's theorem**)

V. Eisler and Z. Zimborás, New Journal of Physics **17**, 053048 (2015).

- Return to Dirac basis (only for particle-number-conserving systems)

Reduced Green's function

$$G_{ij} = \left\langle c_i c_j^\dagger \right\rangle$$
$$G_{A_1,ij} = \left\langle c_i c_j^\dagger \right\rangle$$
with $i \in A_1, j \in A_1$

Gaussian States
(e.g. free-fermion Gibbs state)

$$\rho = \det\left[G\right] e^{\mathbf{c}^\dagger \ln\left(G^{-1}-I\right)\mathbf{c}}$$

Partially transposed Green's function

$$G^{T_2^f} = \begin{pmatrix} G^{11} & \mathrm{i}G^{12} \\ \mathrm{i}G^{21} & I - G^{22} \end{pmatrix}$$

Partial Trace

Fermionic Partial Transpose

$$\rho_{A_1} = \det\left[G_{A_1}\right] e^{\mathbf{c}^\dagger \ln\left(G_{A_1}^{-1}-I\right)\mathbf{c}}$$

Reduced Density Matrix (RDM)

(another)
Gaussian States

$$\rho^{T_2^f} = \det\left[G^{T_2^f}\right] e^{\mathbf{c}^\dagger \ln\left[\left(G^{T_2^f}\right)^{-1}-I\right]\mathbf{c}}$$

Partially Transposed Density Matrix

FPT of interacting fermions within DQMC

- Auxiliary-field QMC: after Trotter decomposition and HS transformation

$$\rho = \sum_{\mathbf{s}} p_{\mathbf{s}} \rho_{\mathbf{s}}$$

$$\rho_{\mathbf{s}} = \det\left[I + e^{W_{\mathbf{s}}}\right]^{-1/2} e^{\frac{1}{4}\boldsymbol{\gamma}^T W_{\mathbf{s}} \boldsymbol{\gamma}} \qquad e^{\widehat{W}_{\mathbf{s}}} = e^{\frac{1}{4}\boldsymbol{\gamma}^T W_{\mathbf{s}} \boldsymbol{\gamma}} \equiv \prod_l e^{\hat{h}_l[\mathbf{s}]},\ e^{\hat{h}_l[\mathbf{s}]} \equiv e^{\hat{K}_l} e^{\hat{V}_l[\mathbf{s}]} e^{\hat{K}_l}$$

by product rule of Gaussian states

$$\rho_{\mathbf{s}} = \det[G_{\mathbf{s}}] e^{\mathbf{c}^\dagger \ln\left[G_{\mathbf{s}}^{-1} - I\right]\mathbf{c}} \qquad e^{\widehat{W}_{\mathbf{s}}} = e^{\mathbf{c}^\dagger W_{\mathbf{s}} \mathbf{c}} \equiv \prod_l e^{\hat{h}_l[\mathbf{s}]},\ G_{\mathbf{s},ij} \equiv \mathrm{Tr}[\rho_{\mathbf{s}} c_i c_j^\dagger] = (I + e^{W_{\mathbf{s}}})^{-1}$$

- After FPT

$$\rho^{T_2^f} = \sum_{\mathbf{s}} p_{\mathbf{s}} \rho_{\mathbf{s}}^{T_2^f}$$

$$\rho_{\mathbf{s}}^{T_2^f} = \rho_0\left[\Gamma_{\mathbf{s}}^{T_2^f}\right],\ \Gamma_{\mathbf{s}}^{T_2^f} = \begin{pmatrix} \Gamma_{\mathbf{s}}^{11} & \mathrm{i}\Gamma_{\mathbf{s}}^{12} \\ \mathrm{i}\Gamma_{\mathbf{s}}^{21} & -\Gamma_{\mathbf{s}}^{22} \end{pmatrix}$$

sign-problem-free → Majorana QMC

Z.-X. Li *et al.*, Phys. Rev. B **91**, 241117 (2015)

sign problematic → Pfaffian QMC

Z.-Y. Han *et al.*, arXiv:2408.10311.

$$\rho_{\mathbf{s}}^{T_2^f} = \rho_0\left[G_{\mathbf{s}}^{T_2^f}\right],\ G_{\mathbf{s}}^{T_2^f} = \begin{pmatrix} G_{\mathbf{s}}^{11} & \mathrm{i}G_{\mathbf{s}}^{12} \\ \mathrm{i}G_{\mathbf{s}}^{21} & I_2 - G_{\mathbf{s}}^{22} \end{pmatrix}$$

Determinant QMC (DQMC)

D. J. Scalapino and R. L. Sugar, Phys. Rev. B **24**, 4295 (1981).
R. Blankenbecler *et al.*, Phys. Rev. D **24**, 2278 (1981).
J. E. Hirsch, Phys. Rev. B **31**, 4403 (1985).

Rényi negativity (RN) and Grover determinants

- The trace of moments of PTDM via replica trick

$$e^{(1-r)\mathcal{E}_r} = \mathrm{Tr}\left[\left(\rho^{T_2^f}\right)^r\right] = \sum_{\mathbf{s}^{(1)}\ldots\mathbf{s}^{(r)}} \left(\prod_{i=1}^{r} p_{\mathbf{s}^{(i)}}\right)\det g_{r,\bar{\mathbf{s}}}$$

$\mathbf{s}^{(\alpha)}$: HS auxiliary fields in replica $\alpha \in [1, r]$

rank-r **Grover determinant**: $\det g_{\mathrm{r},\bar{\mathbf{s}}} = \det\left[\prod_{i=r}^{1} G_{\mathbf{s}^{(i)}}^{T_2^f}\right]\det\left[I + \prod_{i=r}^{1}\left(\left(G_{\mathbf{s}^{(i)}}^{T_2^f}\right)^{-1} - I\right)\right]$

❖ $r = 2$: $\det g_{2,\bar{\mathbf{s}}} = \det\left[G_{\mathbf{s}^{(1)}}^{T_2^f} G_{\mathbf{s}^{(2)}}^{T_2^f} + \left(I - G_{\mathbf{s}^{(1)}}^{T_2^f}\right)\left(I - G_{\mathbf{s}^{(2)}}^{T_2^f}\right)\right]$ (without invoking inverse of $G_{\mathbf{s}}^{T_2^f}$)

- Sign-problem of the Grover determinants: sufficient conditions for $\det\left[g_{r,\bar{\mathbf{s}}}\right] \geq 0$ (necessary for designing incremental algorithms)

(1) $G_{ij}^{\downarrow} = (-1)^{i+j}(\delta_{ij} - G_{ji}^{\uparrow *})$ (e.g. half-filled Hubbard model on bipartite lattices)

(2) $\Gamma_{ij}^{(2)} = (-1)^{i+j}\Gamma_{ij}^{(1)*}$(e.g. half-filled spinless t-V model on bipartite lattices)

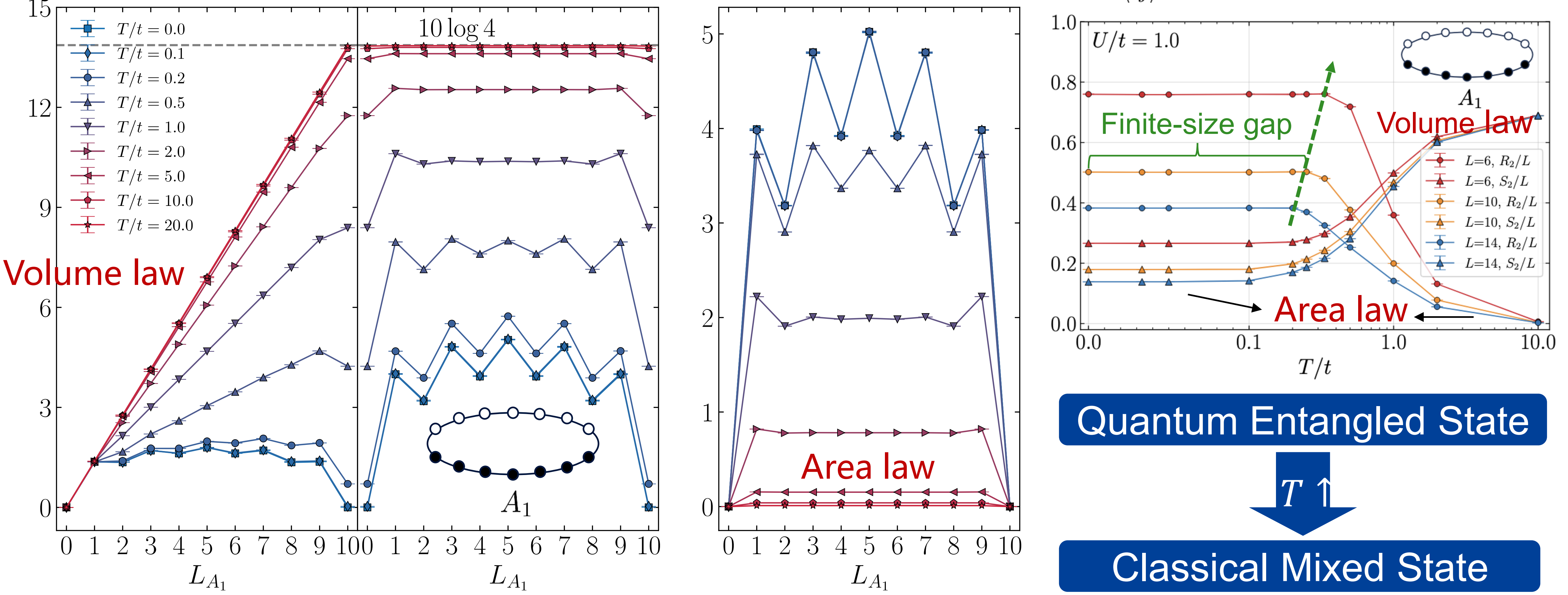
Contents of Part I

Hubbard model: quantum-classical crossover

$$S_2 = -\ln[\mathrm{Tr}(\rho_{A_1})^2] \quad \mathcal{E}_2 = -\ln[\mathrm{Tr}(\rho^{T_2^f})^2] \quad R_2 = -\ln[\frac{\mathrm{Tr}(\rho^{T_2^f})^2}{\mathrm{Tr}(\rho)^2}] = \mathcal{E}_2 - S_2^{\mathrm{th}}$$

$$H = -t\sum_{\langle ij\rangle\sigma}(c_{i\sigma}^\dagger c_{j\sigma} + \mathrm{H.c.}) + \frac{U}{2}\sum_i (n_i - 1)^2$$

Rényi EE | Rényi negativity | Rényi negativity ratio



Quantum Entangled State

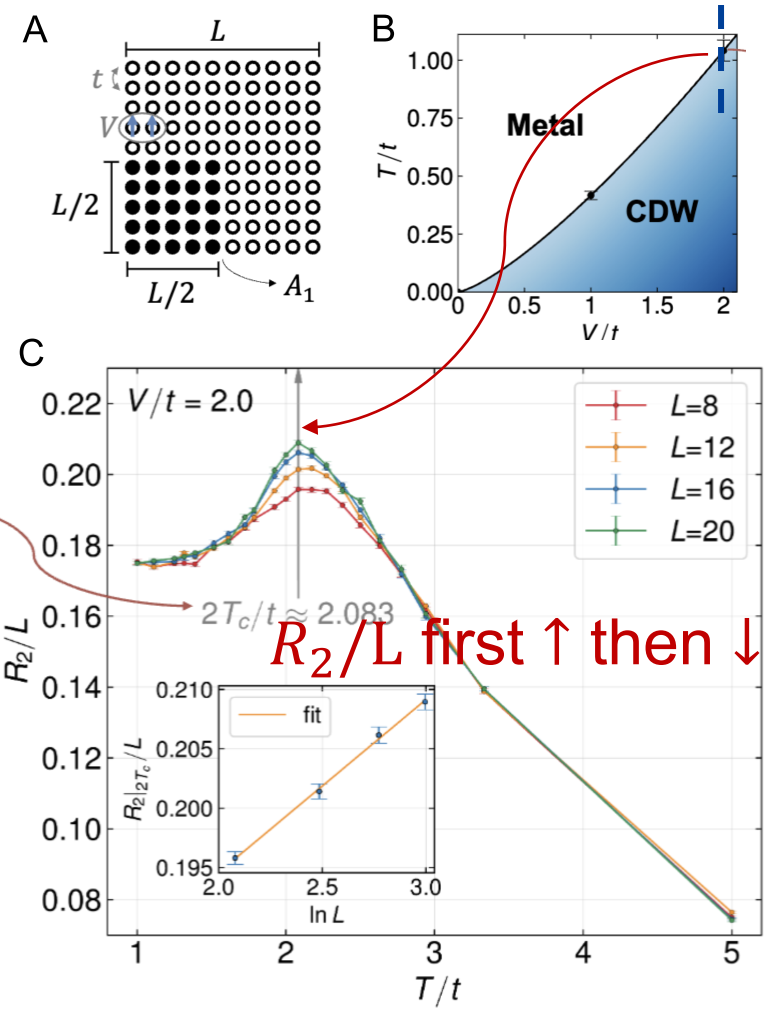
$T \uparrow$

Classical Mixed State

Spinless t-V model: Finite-T transition

$$H = -t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) + V \sum_{\langle i,j \rangle} \left(n_i - \frac{1}{2}\right)\left(n_j - \frac{1}{2}\right)$$ sign-problem-free

Z.-X. Li et al., Phys. Rev. B 91, 241117 (2015).
E. F. Huffman and S. Chandrasekharan, Phys. Rev. B 89, 111101 (2014).
Z. C. Wei et al., Phys. Rev. Lett. 116, 250601 (2016).
L. Wang et al., Phys. Rev. Lett. 115, 250601 (2015).

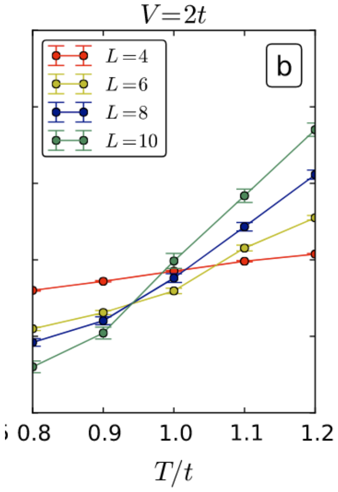


R_2/L first ↑ then ↓

- ➢ qualitatively very different from the **bosonic case**

- ❑ Away from the critical point
  —— Area law
- ❑ Around the critical point
  —— beyond-area-law
- ❑ At the critical point
  —— $R_2 \sim L \ln L$

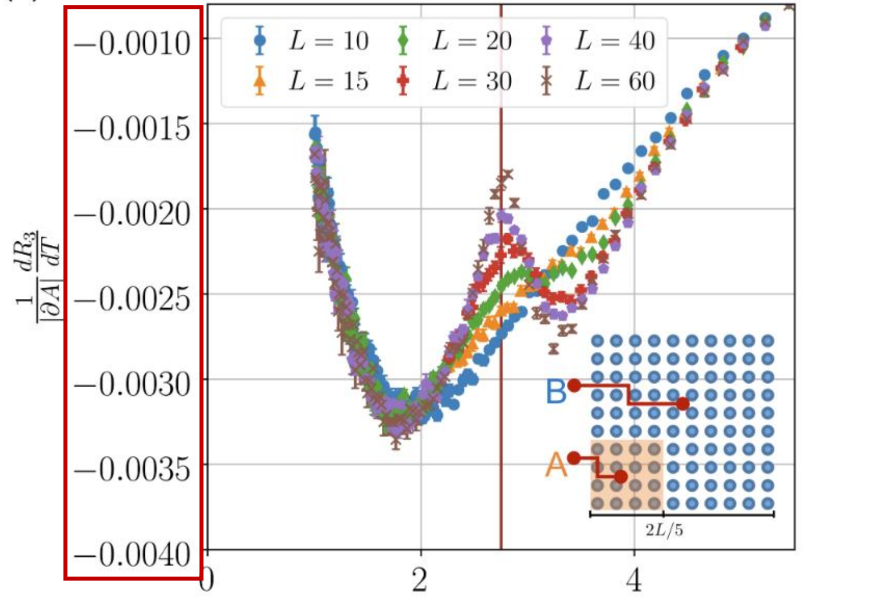
- ➢ different from the behavior of **mutual information** (intersection)



$$H = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h_x \sum_i \sigma_i^x,$$

$$R_3(A) = -\log\left(\frac{\mathrm{tr}\{(\rho^{T_A})^3\}}{\mathrm{tr}\rho^3}\right) = -\log\left(\frac{Z[A,\beta,3]}{Z[3\beta]}\right),$$

derivative of $R_3 < 0$, i.e., $R_3 \downarrow$



Contents of Part I

Twisted PT vs untwisted PT

$$\left(|n_{A_1}, n_{A_2}\rangle\langle\bar{n}_{A_1}, \bar{n}_{A_2}|\right)^{T_2^f} = (-1)^{\phi(n,\overline{n})} |n_{A_1}, \bar{n}_{A_2}\rangle\langle\bar{n}_{A_1}, n_{A_2}|$$

Untwisted PT:
$$\phi(n, \overline{n}) = \frac{[(\tau_2 + \bar{\tau}_2) \bmod 2]}{2} + (\tau_1 + \bar{\tau}_1)(\tau_2 + \bar{\tau}_2)$$

Pseudo-Hermitian: $\rho^{T_2^f \dagger} = X_2 \rho^{T_2^f} X_2, \ \text{where} \ X_2 = (-1)^{\sum_{j\in A_2} c_j^\dagger c_j}$

Twisted PT:
$$\tilde{\phi}(n, \overline{n}) = \phi(n, \overline{n}) + \tau_2$$

Hermitian: $\left(\rho^{\tilde{T}_2^f}\right)^\dagger = \rho^{\tilde{T}_2^f}$

Interesting properties:

If there is a parity constrain: $\mathrm{Tr}\left[\left(\rho^{\tilde{T}_2^f}\right)^{4r}\right] = \mathrm{Tr}\left[\left(\rho^{T_2^f}\right)^{4r}\right], r \in \mathbb{Z}.$

Twisted Rényi negativity vs untwisted Rényi negativity

Twisted PT
Green's function

$$G^{\tilde{T}_2^f} = \begin{pmatrix} I_1 & 2G_{12}^{T_2^f} \\ 0 & 2G_{22}^{T_2^f} - I_2 \end{pmatrix}^{-1} G^{T_2^f}$$

Gaussian States
(e.g. free-fermion Gibbs state)

$$\rho = \det[G]\, e^{\mathbf{c}^\dagger \ln(G^{-1} - I)\mathbf{c}}$$

Untwisted PT
Green's function

$$G^{T_2^f} = \begin{pmatrix} G^{11} & \mathrm{i}G^{12} \\ \mathrm{i}G^{21} & I - G^{22} \end{pmatrix}$$

Twisted PT | Untwisted PT

$$\rho^{\tilde{T}_2^f} = \det\left[G^{\tilde{T}_2^f}\right] e^{\mathbf{c}^\dagger \ln\left[\left(G^{\tilde{T}_2^f}\right)^{-1} - I\right]\mathbf{c}}$$

Twisted PT Density Matrix

(another)
Gaussian States

$$\rho^{T_2^f} = \det\left[G^{T_2^f}\right] e^{\mathbf{c}^\dagger \ln\left[\left(G^{T_2^f}\right)^{-1} - I\right]\mathbf{c}}$$

Untwisted PT Density Matrix

Twisted RN

$$\tilde{\mathcal{E}}_2 = -\ln \mathrm{Tr}[\rho^2]$$

Rank-2 is trivial, high rank needed

Untwisted RN

$$\mathcal{E}_2 = -\ln \mathrm{Tr}[(\rho X_2)^2]$$

Rank-2 is non-trivial

Contents of Part I

Difficulty 1: Singularity of Green's functions

Grover determinant: $\det g_{r,\bar{\mathbf{s}}} = \det\left[\prod_{i=r}^{1} G_{\mathbf{s}(i)}^{T_2^f}\right] \det\left[I + \prod_{i=r}^{1} \mathbb{B}_i\right]$, with $\mathbb{B}_i \equiv (G_{\mathbf{s}(i)}^{T_2^f})^{-1} - I$

DQMC weight $w_{\mathbf{s}} = \det[I + B_{\mathbf{s}}(\beta, 0)]$, with $B_{\mathbf{s}}(\beta, 0) = G_{\mathbf{s}}^{-1} - I$

- Numerical stable formula for $\det g_{r\bar{\mathbf{s}}}$:

$$\ln \det g_{r,\bar{\mathbf{s}}} = \sum_{i=1}^{r} \ln \det G_{\mathbf{s}(i)}^{T_2^f} + \ln \det U + \ln \det D_+$$
$$+ \ln \det\left(D_+^{-1} U^\dagger + D\_V\right).$$

- UDV decomp. $\mathbb{B}(r, 0) \equiv \prod_{j=r}^{1} \mathbb{B}_j = UDV$
- Stable inverse

$$\mathbb{G} \equiv \left(I + \mathbb{B}(r, 0)\right)^{-1} = \left[D_+^{-1} U^\dagger + D\_V\right]^{-1} D_+^{-1} U^\dagger$$

with $D_+ = \max(D, 1)$, $D_- = \min(D, 1)$

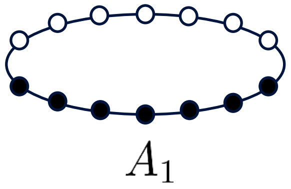
- Sims: matrices with exponentially large conditional number
- Diffs: $\mathbb{B}_i$ do not know from a priori, have to be evaluate by inversing $G_{\mathbf{s}(i)}^{T_2^f}$.
  - Unlike $B_{\mathbf{s}}(\beta, 0) = \prod_l e^{K_l} e^{V[s_l]}$

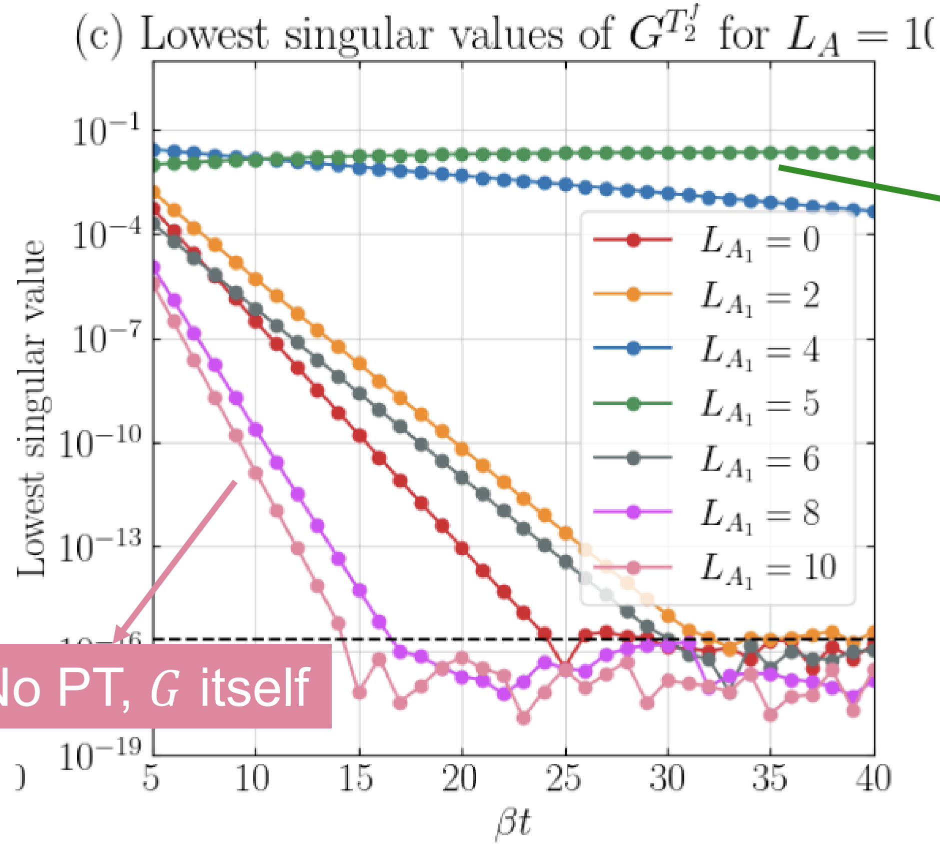
$\Rightarrow$ If $\mathbb{B}_i$ have already lost information, the stable formula still fails.

F.-H. Wang and X. Y. Xu, 2503.07731. Related discussion on high-rank EE: X. Zhang *et al.*, Phys. Rev. B **109**, 205147 (2024).

Difficulty 1: Singularity of Green's functions

- $G_{\mathbf{s}}^{T_2^f} = \begin{pmatrix} G_{\mathbf{s}}^{11} & \mathrm{i}G_{\mathbf{s}}^{12} \\ \mathrm{i}G_{\mathbf{s}}^{21} & I_2 - G_{\mathbf{s}}^{22} \end{pmatrix}$ inherits **singularity** from $G_{\mathbf{s}}$
- illustrated in a free-fermion chain $H = -t\sum_i \left(c_i^\dagger c_{i+1} + \mathrm{H.c.}\right) - \mu \sum_i c_i^\dagger c_i$





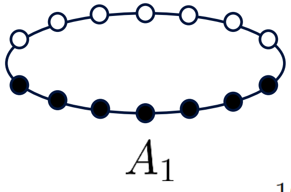
- Less singular than Green's function in general
- Depend on ***bipartition geometry***
  - At equal-bipartition point ($L_{A_1} = L_A/2$) is non-singular.
  - Less singular as L_{A_1} close to equal-bipartition point.
- Depend on ***temperature***
  - $\sigma_{\min} \sim \exp(-c\beta t)$, exponentially decrease as temperature decreases
- Depend on *model and HS transformation*

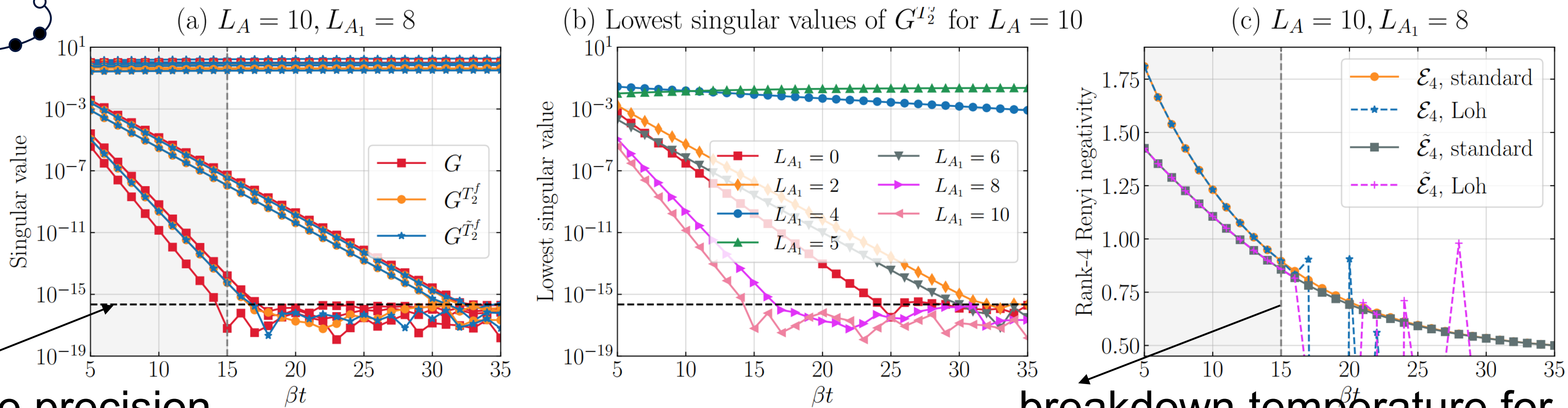
Difficulty 1: Singularity of Green's functions

Grover determinant: $\det g_{r,\bar{\mathbf{s}}} = \det\left[\prod_{i=r}^{1} G_{\mathbf{s}(i)}^{T_2^f}\right] \det[I + \mathbb{B}(r,0)]$, with $\mathbb{B}(r,0) \equiv \prod_{j=r}^{1} \mathbb{B}_j = UDV$

❖ Stable formula: $\ln\det g_{r,\bar{\mathbf{s}}} = \sum_{i=1}^{r} \ln\det G_{\mathbf{s}(i)}^{T_2^f} + \ln\det U + \ln\det D_+ + \ln\det\left(D_+^{-1}U^\dagger + D_{-}V\right)$

❖ Standard formula (only for free fermions): $\det g_r = \det\left[\left(G^{T_2^f}\right)^r + \left(I - G^{T_2^f}\right)^r\right]$





machine precision of Float64

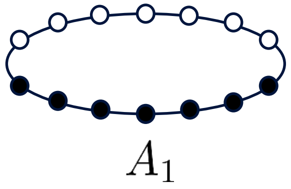
Regularization is required!

breakdown temperature for $L_{A_1} = 8$

Difficulty 1: Singularity of Green's functions

Grover determinant: $\det g_{r,\bar{s}} = \det\left[\prod_{i=r}^{1} G_{\mathbf{s}^{(i)}}^{T_2^f}\right] \det[I + \mathbb{B}(r,0)]$, with $\mathbb{B}(r,0) \equiv \prod_{j=r}^{1} \mathbb{B}_j = UDV$

❖ Solution 2: Drut's reconstruction of the Grover matrix

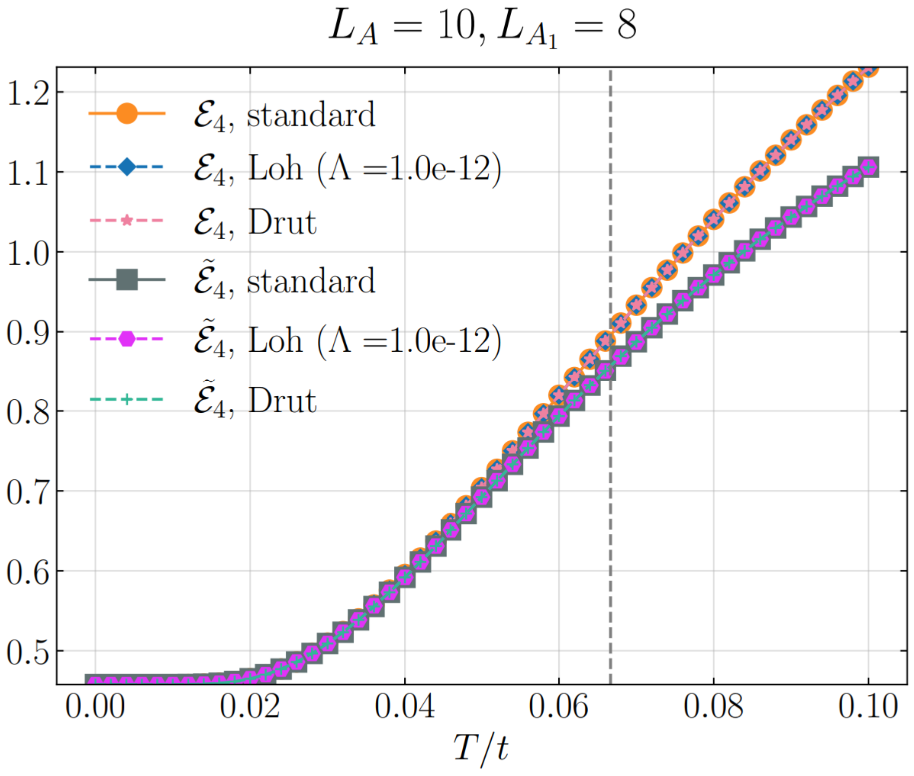


$$\det g_{r,\bar{\mathbf{s}}} = \det \mathcal{G}_{1\to r,\bar{\mathbf{s}}}^{T_2^f} \det K_{r,\bar{\mathbf{s}}} = \det T_{r,\bar{\mathbf{s}}}$$

$$K_{r,\bar{\mathbf{s}}} \equiv \begin{pmatrix} I & 0 & \cdots & 0 & \mathbb{B}_{\mathbf{s}^{(r)}} \\ -\mathbb{B}_{\mathbf{s}^{(1)}} & I & \cdots & 0 & 0 \\ 0 & -\mathbb{B}_{\mathbf{s}^{(2)}} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\mathbb{B}_{\mathbf{s}^{(r-1)}} & I \end{pmatrix}$$

$$T_{r,\bar{\mathbf{s}}} \equiv K_{r,\bar{\mathbf{s}}} \mathcal{G}_{1\to r,\bar{\mathbf{s}}}^{T_2^f}$$

$$= \begin{pmatrix} G_{\mathbf{s}^{(1)}}^{T_2^f} & 0 & \cdots & 0 & I - G_{\mathbf{s}^{(r)}}^{T_2^f} \\ G_{\mathbf{s}^{(1)}}^{T_2^f} - I & G_{\mathbf{s}^{(2)}}^{T_2^f} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & G_{\mathbf{s}^{(r-1)}}^{T_2^f} & 0 \\ 0 & 0 & \cdots & G_{\mathbf{s}^{(r-1)}}^{T_2^f} - I & G_{\mathbf{s}^{(r)}}^{T_2^f} \end{pmatrix}$$



Difficulty 2: exponentially large variance

Incremental algorithm for Rényi negativity

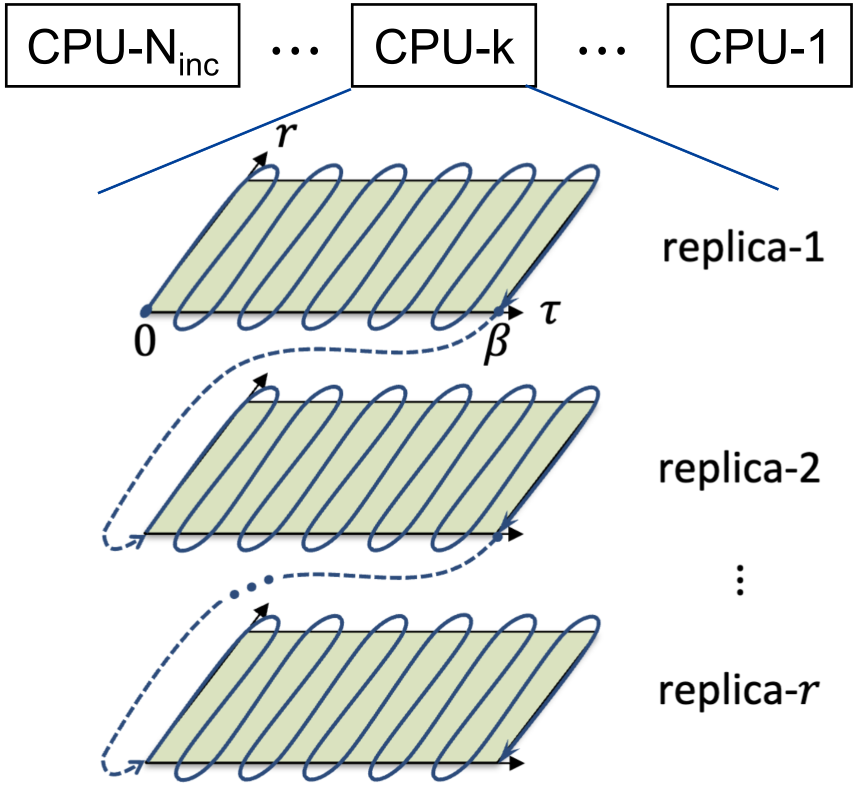
Y. D. Liao, arXiv:2307.10602.
Y. D. Liao *et al.*, arXiv:2302.11742.
J. D'Emidio *et al.*, Phys. Rev. Lett. **132**, 076502 (2024).
X. Zhang *et al.*, Phys. Rev. B **109**, 205147 (2024).

$$Z_k = \sum_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} W_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} \left(\det g_{r,\bar{\mathbf{s}}}\right)^{\frac{k}{N_{\mathrm{inc}}}}$$

$$e^{(1-r)\mathcal{E}_r} = \frac{\sum_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} W_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} \det g_{r,\bar{\mathbf{s}}}}{\sum_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} W_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}}} = \frac{Z_{N_{\mathrm{inc}}}}{Z_{N_{\mathrm{inc}}-1}} \cdots \frac{Z_{k+1}}{Z_k} \cdots \frac{Z_1}{Z_0}$$

$$\frac{Z_{k+1}}{Z_k} = \frac{\sum_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} W^{(k)}_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} \left(\det g_{r,\bar{\mathbf{s}}}\right)^{\frac{1}{N_{\mathrm{inc}}}}}{\sum_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} W^{(k)}_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}}} = \left\langle \left(\det g_{r,\bar{\mathbf{s}}}\right)^{\frac{1}{N_{\mathrm{inc}}}} \right\rangle_{W^{(k)}}$$

joint weight $W^{(k)}_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} \equiv W_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(r)}} \det g_{r,\bar{\mathbf{s}}}$



Forward local sweep

Acceptence ratio for k-th factor (update at replica-i)

$$R_i^{(k)} \equiv \frac{W^{(k)}_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(i)\prime}\dots\mathbf{s}^{(r)}}}{W^{(k)}_{\mathbf{s}^{(1)}\dots\mathbf{s}^{(i)}\dots\mathbf{s}^{(r)}}} = \frac{w_{\mathbf{s}^{(i)\prime}}}{w_{\mathbf{s}^{(i)}}} \left(\frac{\det g_{r,\bar{\mathbf{s}}'}}{\det g_{r,\bar{\mathbf{s}}}}\right)^{k/N_{\mathrm{inc}}}$$

Difficulty 2: exponentially large variance

Acceptence ratio for k-th factor (update at replica-i) $R_i^{(k)} \equiv \frac{W^{(k)}_{\mathbf{s}^{(1)}\ldots\mathbf{s}^{(i)\prime}\ldots\mathbf{s}^{(r)}}}{W^{(k)}_{\mathbf{s}^{(1)}\ldots\mathbf{s}^{(i)}\ldots\mathbf{s}^{(r)}}} = \frac{w_{\mathbf{s}^{(i)\prime}}}{w_{\mathbf{s}^{(i)}}}\left(\frac{\det g_{r,\bar{\mathbf{s}}'}}{\det g_{r,\bar{\mathbf{s}}}}\right)^{k/N_{\mathrm{inc}}}$

+ Difficulty 1: need a stable object suitable for doing fast update

$$\mathbb{F}_i = \left[G^{T_2^f}_{\mathbf{s}^{(i)}}\mathbb{B}_{\setminus i}^{-1}G^{T_2^f}_{\mathbf{s}^{(i-1)}}\mathbb{B}_{i-1} + G^{T_2^f}_{\mathbf{s}^{(i)}}\mathbb{B}_i G^{T_2^f}_{\mathbf{s}^{(i-1)}}\mathbb{B}_{i-1}\right]^{-1} = \left[G^{T_2^f}_{\mathbf{s}^{(i)}}\mathbb{F}_{\setminus i}^{-1} + I - G^{T_2^f}_{\mathbf{s}^{(i-1)}}\right]^{-1}$$

$$\mathbb{B}_{\setminus i} \equiv \mathbb{B}(i-1,0)\,\mathbb{B}(r,i) = \mathbb{B}_{i-1}\cdots\mathbb{B}_1\mathbb{B}_r\cdots\mathbb{B}_{i+1}$$

$$\mathbb{F}_{\setminus i}^{-1} \equiv \left(\mathbb{B}_{\setminus i}^{-1} - I\right)G^{T_2^f}_{\mathbf{s}^{(i-1)}}\mathbb{B}_{i-1}$$

$$= \boxed{\mathbb{B}_{i+1}^{-1}\cdots\mathbb{B}_r^{-1}\mathbb{B}_1^{-1}\cdots\mathbb{B}_{i-2}^{-1}}\,G^{T_2^f}_{\mathbf{s}^{(i-1)}} - \left(I - G^{T_2^f}_{\mathbf{s}^{(i-1)}}\right)$$

$r = 2$

$$\mathbb{F}_i^{-1} = G^{T_2^f}_{\mathbf{s}^{(i)}}G^{T_2^f}_{\mathbf{s}^{(i-1)}} + \left(I - G^{T_2^f}_{\mathbf{s}^{(i)}}\right)\left(I - G^{T_2^f}_{\mathbf{s}^{(i-1)}}\right) = g_{2,\bar{\mathbf{s}}}$$

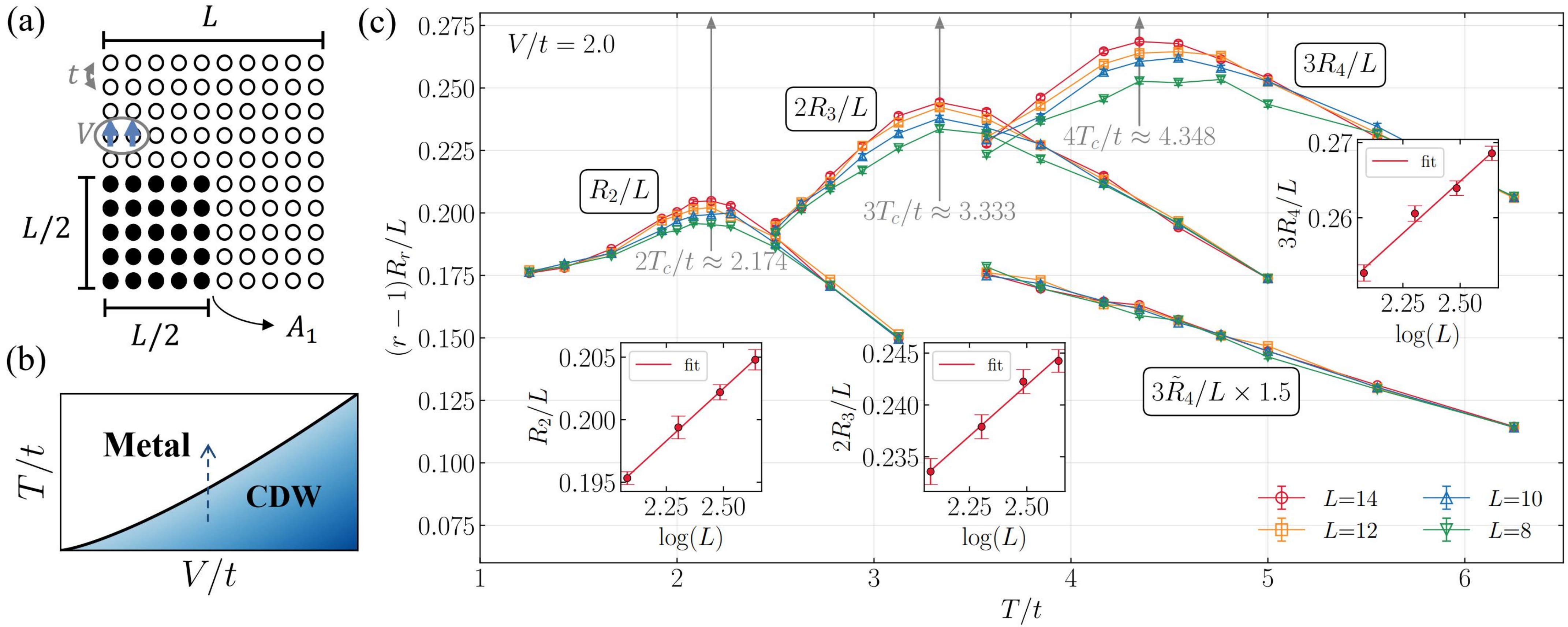
Updating the inverse of Grover matrix is common in the previous studies on EE

- Update ratio: $\frac{\det g_{r,\bar{\mathbf{s}}'}}{\det g_{r,\bar{\mathbf{s}}}} = \frac{\det \mathbb{F}_i}{\det \mathbb{F}_i'}$
- Fast update √
- More stable due to
  - working with $\mathbb{B}_i^{-1}$ instead of $\mathbb{B}_i$
  - involve $G^{T_2^f}_{\mathbf{s}^{(i-1)}}\mathbb{B}_{i-1} = (I - G^{T_2^f}_{\mathbf{s}^{(i-1)}})$ to cancels one additional $\left(G^{T_2^f}_{\mathbf{s}^{(i-1)}}\right)^{-1}$.

Revisit the finite-T transition in t-V model

$$H = -t\sum_{\langle i,j\rangle}(c_i^\dagger c_j + c_j^\dagger c_i) + V\sum_{\langle i,j\rangle}\left(n_i - \frac{1}{2}\right)\left(n_j - \frac{1}{2}\right)$$

$$(1-r)R_r = \ln\frac{\mathrm{Tr}\left[\left(\rho^{T_2^f}\right)^{\mathrm{r}}\right]}{\mathrm{Tr}[\rho^r]} \qquad (1-r)\tilde{R}_r = \ln\frac{\mathrm{Tr}\left[\left(\rho^{\tilde{T}_2^f}\right)^{\mathrm{r}}\right]}{\mathrm{Tr}[\rho^r]}$$



Contents of Part I

1. Introduction: Mixed-state entanglement and negativity
2. Fermionic partial transpose: From free fermions to interacting fermions
3. Temperature-dependence of Rényi negativity in strongly-correlated fermionic models
4. Twisted Rényi negativity vs untwisted Rényi negativity
5. Stable computation of high-rank Rényi negativities
6. Summary

Summary

- Gaussian-preserving property of the fermionic partial transpose makes it possible to write partially transposed density matrix as a weighted sum of Gaussian state.
- The Rényi negativity can be sampled within DQMC framework via replica trick, with the estimators being the Grover determinants.
  - Grover determinants have been proved to be real and non-negative for two classes of sign-problem free models.
- Untwisted Rényi negativity ratio exhibits beyond-area-law behavior around the finite-temperature transition point in the spinless t-V model.
- Twisted Rényi negativity obeys area-law, thus, a better Rényi proxy of LN
- High-rank Rényi negativity can be stably calculated.

Outline

Part I: Entanglement negativity

- Introduction: Mixed-state entanglement and negativity
- Fermionic partial transpose: From free fermions to interacting fermions
- Temperature-dependence of Rényi negativity in strongly-correlated fermionic models
- Twisted Rényi negativity vs untwisted Rényi negativity
- Stable computation of high-rank Rényi negativities
- Summary

Part II: From entanglement to magic

- Stabilizer Rényi entropy (SRE)
- two-point SRE for interacting fermion systems
- Summary

From entanglement to magic

- **Entanglement:** Quantum correlations
  - Distinguishes quantum from classical states
  - Resource for quantum communication
- **Magic:** Non-stabilizerness
  - Distinguishes universal quantum states from efficiently simulable ones
  - Resource for quantum computation
  - Stabilizer states: efficiently simulable (Gottesman-Knill theorem)
- **Question:** Can we compute magic in interacting fermions?
  - Exponential wall of Hilbert space
  - Exponential wall of Pauli strings (majorana strings)

Stabilizer Rényi entropy (SRE)

- **Definition:** Measure of distance from stabilizer states

$$M_\alpha(\rho) = \frac{1}{1-\alpha} \ln \sum_P \mathrm{Tr}[P\rho]^{2\alpha}$$

L Leone, SFE Oliviero, A Hamma, Phys. Rev. Lett. 128, 050402 (2022).

  - P: Pauli strings (stabilizer group elements)
  - $M_\alpha = 0$ for stabilizer states
  - $M_\alpha > 0$ for non-stabilizer (magical) states
- **Problem:** double exponential walls
  - ρ: the density matrix of interacting systems
  - P: exponentially many Pauli strings
- **Solution:** Monte Carlo or Local magic
  - Restrict to few-body subsystems
  - e.g. two-point SRE (SRE non-increasing under partial trace?)

CD White, C Cao, B Swingle, PRB 103, 075145 (2021)
D Qian, J Wang, PRA 111, 052443 (2025).
LYN Liu, S Yi, J Cui, arXiv:2508.03534

Stabilizer Rényi entropy (SRE)

- **Less studied in fermion system**

$$M_\alpha(\rho) = \frac{1}{1-\alpha} \ln \sum_{\gamma^x} \mathrm{Tr}[\gamma^x \rho]^{2\alpha}$$

  - γ^x: Majorana strings
  - For fermionic Gaussian state:

    SFE Oliviero, L Leone, A Hamma, Phys. Rev. A 106, 042426 (2022).

    M Collura, JD Nardis, V Alba, G Lami, Quantum 10, 2036 (2026)
  - For SYK model:

    S Bera, M Schirò, SciPost Phys. 19, 159 (2025)

    P Zhang, S Zhou, N Sun, arXiv:2509.17417

Global SRE vs Local SRE for interacting fermion

Approach 1:
Two-level MC

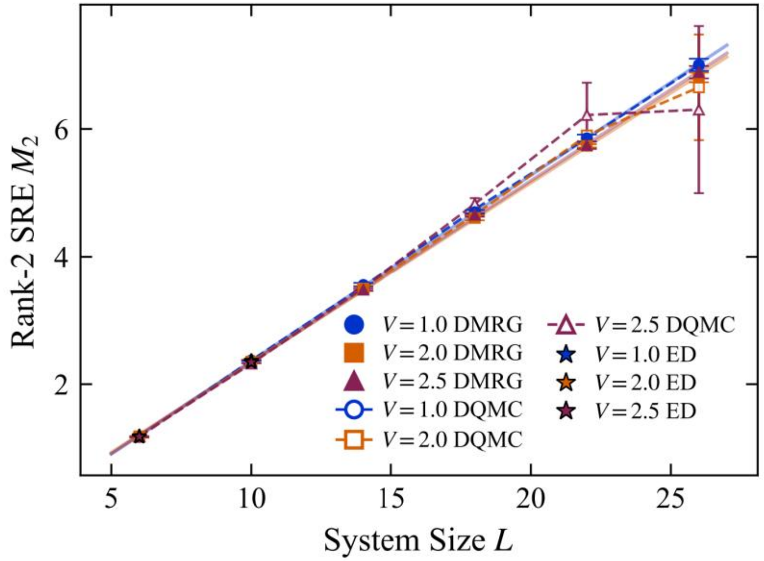
$$M_\alpha(\rho) = \frac{1}{1-\alpha}\log\sum_{P\in\mathcal{P}_N}\frac{1}{d}\left[\mathrm{Tr}(P\rho)\right]^{2\alpha} - S_\alpha(\rho)$$

Inner MC: sample auxiliary fields

$$\mathrm{Tr}(\rho\gamma^{\boldsymbol{x}}) = \sum_{\boldsymbol{s}} P_{\boldsymbol{s}}\mathrm{Tr}(\rho_{\boldsymbol{s}}\gamma^{\boldsymbol{x}}) = \sum_{\boldsymbol{s}} P_{\boldsymbol{s}}\mathrm{Pf}(\Gamma_{\boldsymbol{s}}|_{\boldsymbol{x}})$$

Outer MC: sample Majorana string

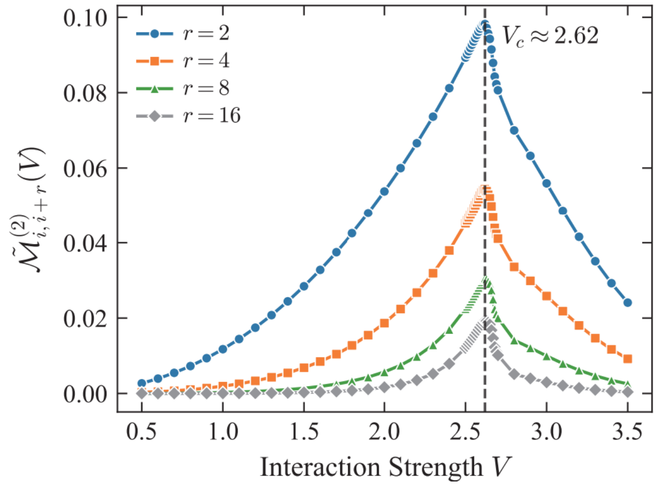
$$\sum_{P\in\mathcal{P}_n}\mathrm{Tr}[P\rho]^{2\alpha} = \sum_{\gamma^{\boldsymbol{x}}}|\mathrm{Tr}[\gamma^{\boldsymbol{x}}\rho]|^{2\alpha}$$



Approach 2:
Tomography
two point reduced density matrix

$$\rho_{i_1 i_2} = \begin{pmatrix} 1-\langle n_{i_1}\rangle - \langle n_{i_2}\rangle + \varrho_r^n & & & \\ & \langle n_{i_2}\rangle - \varrho_r^n & g_r & \\ & g_r^* & \langle n_{i_1}\rangle - \varrho_r^n & \\ & & & \varrho_r^n \end{pmatrix}$$

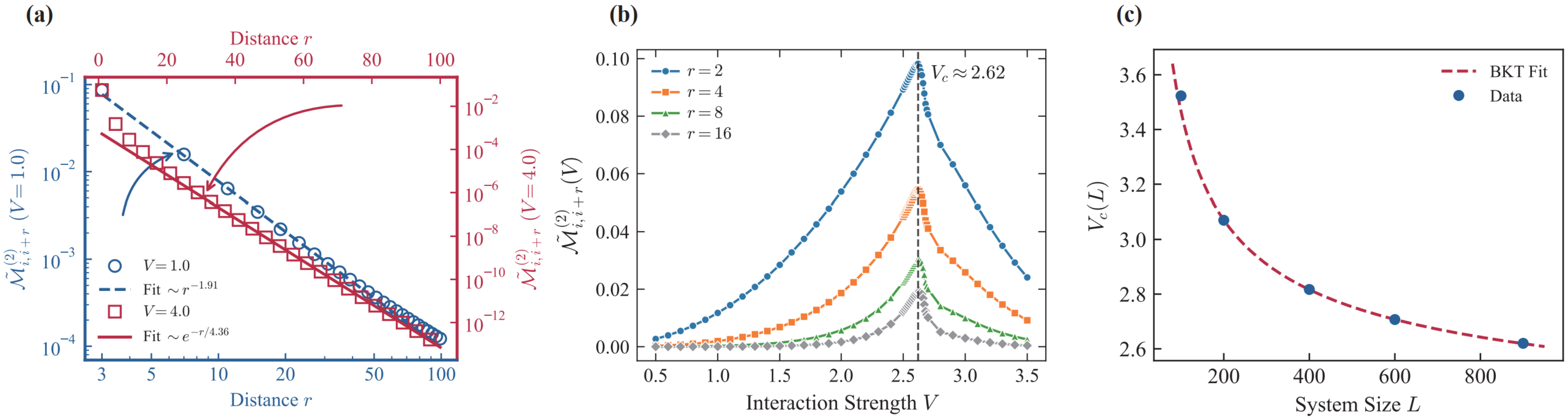
(b)



Fang, Wang, **Xu**, arXiv:2601.13314

Two-point SRE for 1D spinless t-V model

1D spinless t-V model



Mutual two-point SRE

$$\tilde{\mathcal{M}}^{(\alpha)}_{i,j}(\rho) = \mathcal{M}^{(\alpha)}_{i,j}(\rho) - \mathcal{M}^{(\alpha)}_{i}(\rho) - \mathcal{M}^{(\alpha)}_{j}(\rho)$$

$$\xi \sim \exp\left(\pi^2/2\sqrt{|V-V_c|}\right)$$

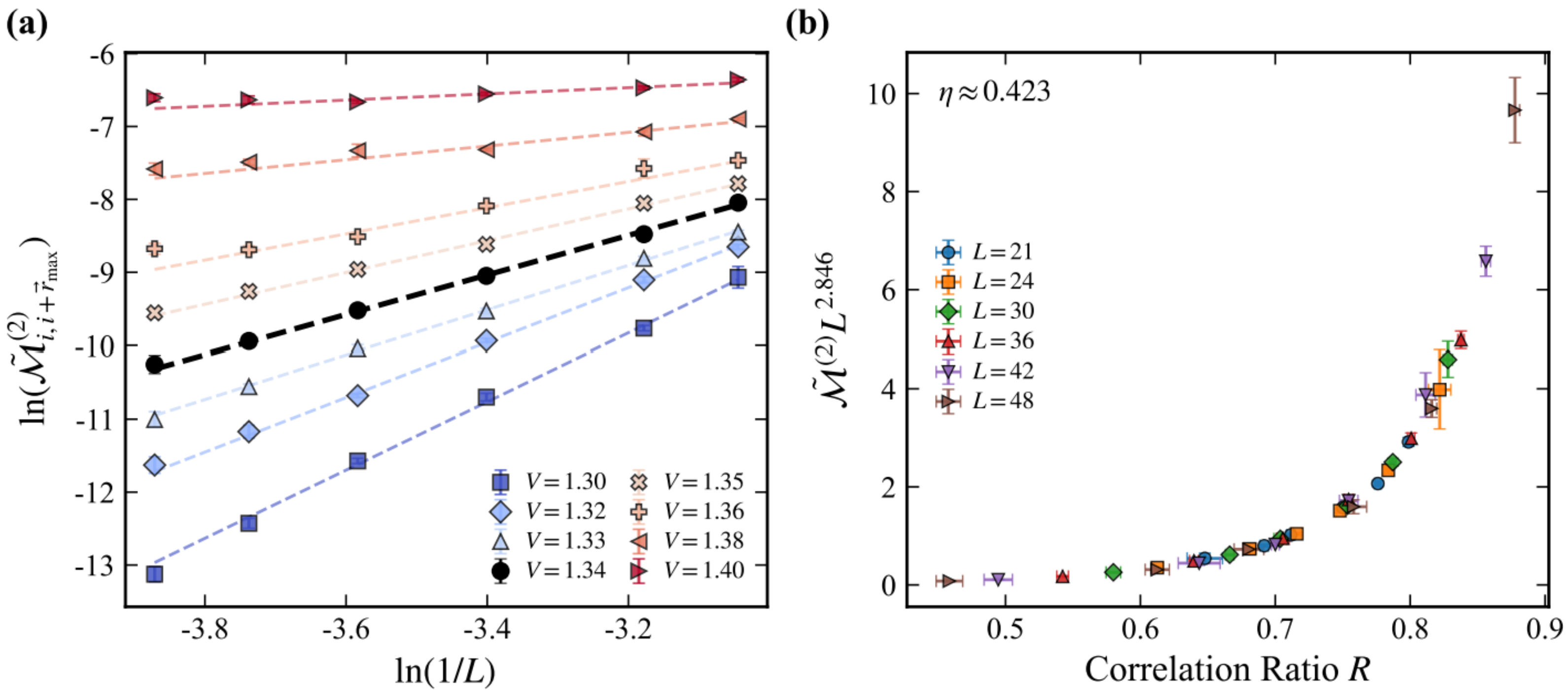
$$V_c(L) = 2 + \frac{\pi^4/4}{(\ln L - \ln L_0)^2}$$

$$V_c(\infty) = 2$$

Fang, Wang, **Xu**, arXiv:2601.13314

Two-point SRE for 2D spinless t-V model

2D spinless t-V model



$$C_{\max}(L) = L^{-1-\eta}\mathcal{F}(L^{1/\nu}(V - V_c))$$

Fang, Wang, **Xu**, arXiv:2601.13314

Two-point SRE for Laughlin state

$\nu = 1/m$ Laughlin state

$$\Psi_m(z_1, z_2, ..., z_N) = \prod_{j<k} (z_j - z_k)^m \exp\left(-\frac{1}{4}\sum_{i=1}^{N} |z_i|^2\right)$$

$$\phi_k(z) \sim z^k e^{-|z|^2/4}$$

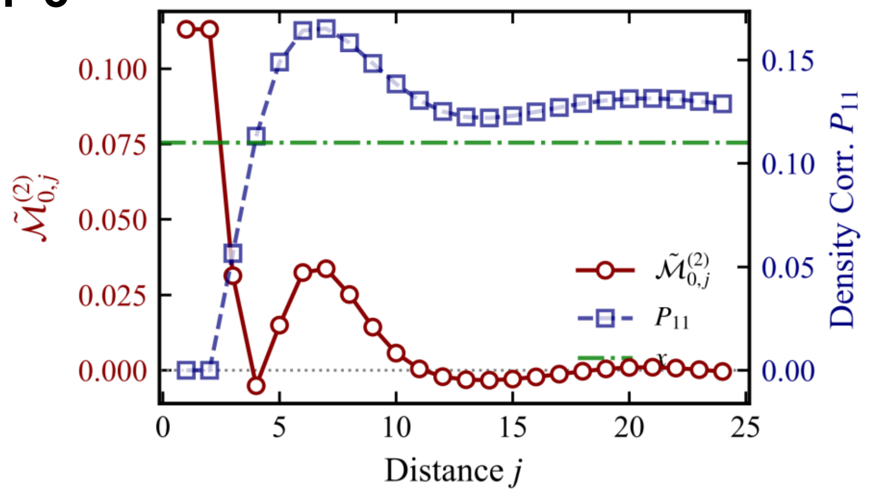
$$\Psi_m(z_1, z_2, ..., z_N) = \sum_{\text{config } \{k_1,\ldots,k_N\}} C^m_{\{k_i\}} \cdot \det[\phi_{k_i}(z_j)].$$

Angular momentum conservation leads to

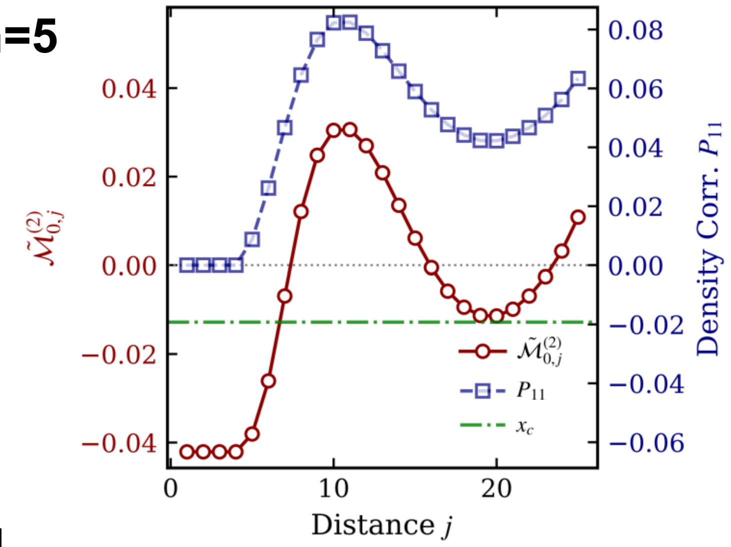
$$\rho_{0j} = \mathrm{diag}(P_{00}, P_{01}, P_{10}, P_{11})$$

$$\begin{pmatrix} 1 - \langle n_{i_1}\rangle - \langle n_{i_2}\rangle + \varrho_r^n & & & \\ & \langle n_{i_2}\rangle - \varrho_r^n & \cancel{g_r} & \\ & \cancel{g_r^*} & \langle n_{i_1}\rangle - \varrho_r^n & \\ & & & \varrho_r^n \end{pmatrix}$$

m=3



m=5



Fang, Wang, **Xu**, arXiv:2601.13314

Summary

- **Two-point SRE**
  - Tractable SRE calculations
  - Tomography from standard correlators (no new measurements)
  - Applicable to DQMC, ED, DMRG, experiments
- **Future directions**
  - Multi-point SRE
  - SRE for dynamical, open or thermal systems
  - Connection to computational complexity

Thank you!