THE UNIVERSITY OF HONG KONG
DEPARTMENT OF PHYSICS

Assignment 1

Course: *Computational Physics (PHYS4150/8150)* – Professor: *Prof. Zi Yang Meng*
Tutor: *Mr. Tim Lok Chau, Mr. Menghan Song*
Due date: *5th October, 2025*

We have learned that the planet moves around the sun in an elliptic orbit and the gravitational force between a planet and the sun is governed by Newton's law of gravitation $\overrightarrow{F} = -\frac{G_N M_{sun} m}{r^2}\frac{\overrightarrow{r}}{r}$. In this assignment, let's discuss Mercury's motion around the sun. You can express the distance and time in the units of $R = 10^{10}$m and $T_0 = 1$ earth day **(i.e. You should set $\mathbf{R_0} = \mathbf{1}$ and $\mathbf{T_0} = \mathbf{1}$ in your code, otherwise marks will be deducted.)**

Below are some constants you would use in assignment 1.

- Schwarzschild radius of Sun $r_s = \frac{2G_N M_{sun}}{c^2} = 2.95 \times 10^{-7} R_0$.
- Square of specific angular momentum $r_L^2 = 8.19 \times 10^{-7} R_0^2$.
- Constant of base acceleration of Mercury $\frac{c^2 r_s}{2} = 0.99 \frac{R_0^3}{T_0^2}$.

1. With the definition of the Schwarzschild radius r_s, rewrite Newton's law of gravitation in the following form,

$$\ddot{\vec{r}} = -\left(\frac{c^2 r_s}{2}\right)\left(\frac{1}{r^2}\right)\frac{\vec{r}}{r} \tag{1}$$

   where c is the speed of light, G_N is the Newtonian constant of gravitation and M_{sun} is the mass of sun. Please show your steps in your Jupyter notebook or submit the steps as a separate PDF.

2. Use Euler method, Leap-Frog method, and 4th-order Runge-Kutta method to computationally solve the equation of motion for the orbit of Mercury. For each method, please use **different value of Δt, in the unit of T_0 (Euler: 3×10^{-4}, 3×10^{-1}, 1 ; Leap Frog: 3×10^{-1}, 1 ; 4th order Runge Kutta: 3×10^{-1}, 1)** (Δt is in the unit of earth day). We consider the system to be in a two-dimensional space. The initial condition for Mercury are set as $x = 0, y = 5.5R_0, v_x = 0.53\frac{R_0}{T_0}, v_y = 0$. Please show the orbit of Mercury in **5 Mercury years** (i.e. 1 Mercury year is equal to 88.0 Earth days). **Plot the orbit in the same graph and compare the performance of different methods and the same method at different Δt.**

3. In reality, Mercury's orbit is not a static ellipse, and the perihelion of Mercury would precess with a small amount every year. It is shown in Fig. 1. [1].

[1]Image source: "https://en.wikipedia.org/wiki/Tests_of_general_relativity#/media/File:Apsidendrehung.png"

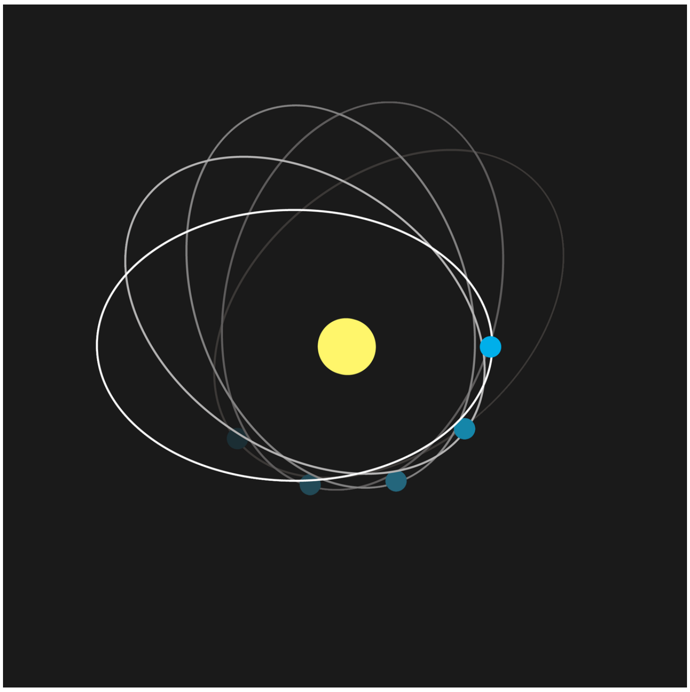

Figure 1: Mercury's perihelion precession

To characterize the correction to Mercury's orbit due to gravitational effects from other planets and the effect of General Relativity, some propose that the correction to the gravitational equation and the acceleration of Mercury should be described as

$$\ddot{\vec{r}} = -\left(\frac{c^2 r_s}{2}\right)\left(\frac{1}{r^2}\right)\left(1+\alpha\frac{r_s}{r}+\beta\frac{r_L^2}{r^2}+\gamma\frac{r_L^4}{r^4}\right)\frac{\vec{r}}{r} \tag{2}$$

Please use the **Euler method and the 4th-order Runge-Kutta method** to solve the equation of motion for Mercury again. Set $\Delta t = 0.05T_0$, $\alpha = \beta = 10^6$, $\gamma = 10^{12}$, time for update to be 750 earth days and $x = 0, y = 5.5R_0, v_x = 0.53\frac{R_0}{T_0}, v_y = 0$. **Draw the trajectory of Mercury and show that the orbit indeed exhibits similar precession as in Fig. 1, compare the performance of the Euler method and 4th-order Runge Kutta in the same plot, and comment on the methods.**

4. **(The initial condition for this question is different from the previous questions)** Set $\Delta t = 0.005T_0$, time for update to be 500 earth days, $\alpha = \gamma = 0$ and the initial condition as $x = 0, y = 4.5R_0, v_x = 0.40\frac{R_0}{T_0}, v_y = 0$, plot the orbit of Mercury with $\beta = 0, 2\times10^4, 4\times10^4, 6\times10^4, 8\times10^4, 10^5$ and shows the perihelion of each revolution in the plot. Fig. 4 below is an example showing the orbit and perihelion in $\beta = 10^5$.

   Then, **repeat question 4 simulation with** $\gamma = 10^{12}$. After all, plot the shift in the perihelion position against the β value and extract the slope. Fig. 4 below is the required plot.

Hint:

- Note that perihelion is the point in the orbit closest to the sun. Putting the sun in the origin, **when** $|\vec{r}_t| < |\vec{r}_{t+1}|$ **and** $|\vec{r}_t| < |\vec{r}_{t-1}|$**,** $|\vec{r}_t|$ **should be the perhelion**, where $\vec{r}_t = x(t)\hat{i} + y(t)\hat{j}$ and $|\vec{r}_t|$ is the magnitude of $\vec{r}_t$.
- The angles between 2 nearest perihelion positions (i.e. the angular shift in perihelion position) can be computed using the following formula,

$$\delta\Theta = \arccos\left(\frac{\vec{r}_{Perhelion_1} \cdot \vec{r}_{Perhelion_2}}{|\vec{r}_{Perhelion_1}||\vec{r}_{Perhelion_2}|}\right) \tag{3}$$

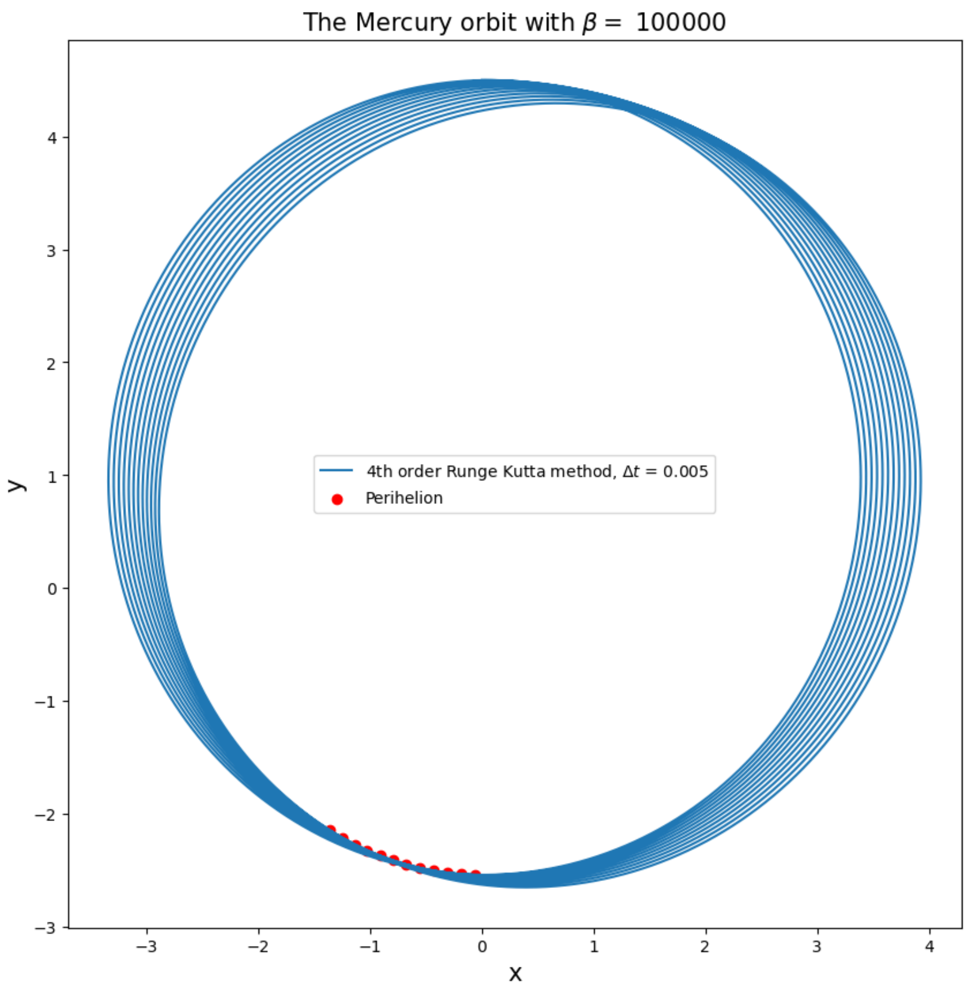

Figure 2: Mercury's perihelion precession and the perihelions in each revolution for $\beta = 10^5$

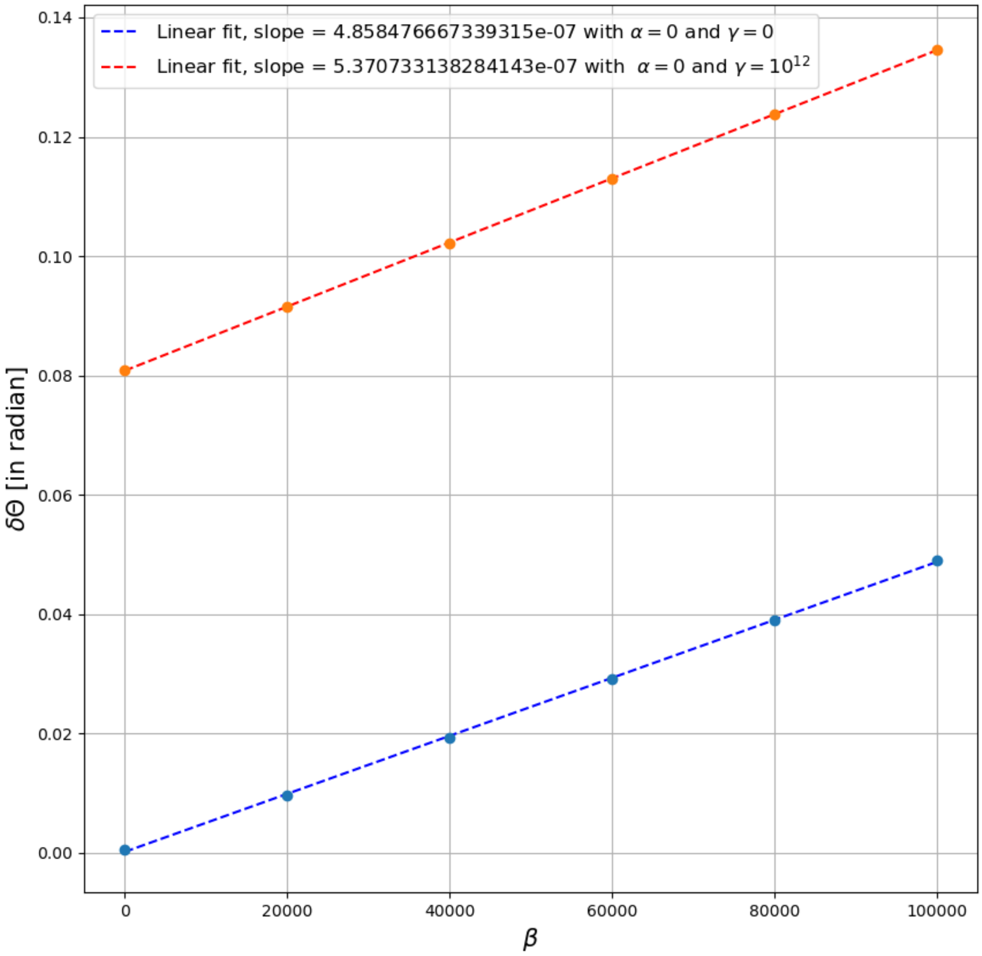

Figure 3: Plot of shift in perihelion position and β