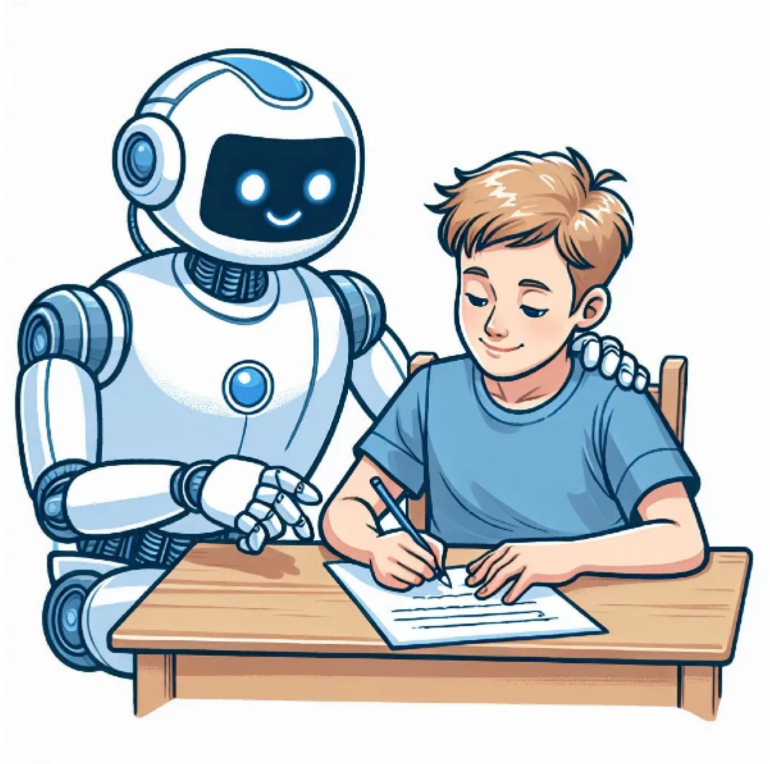
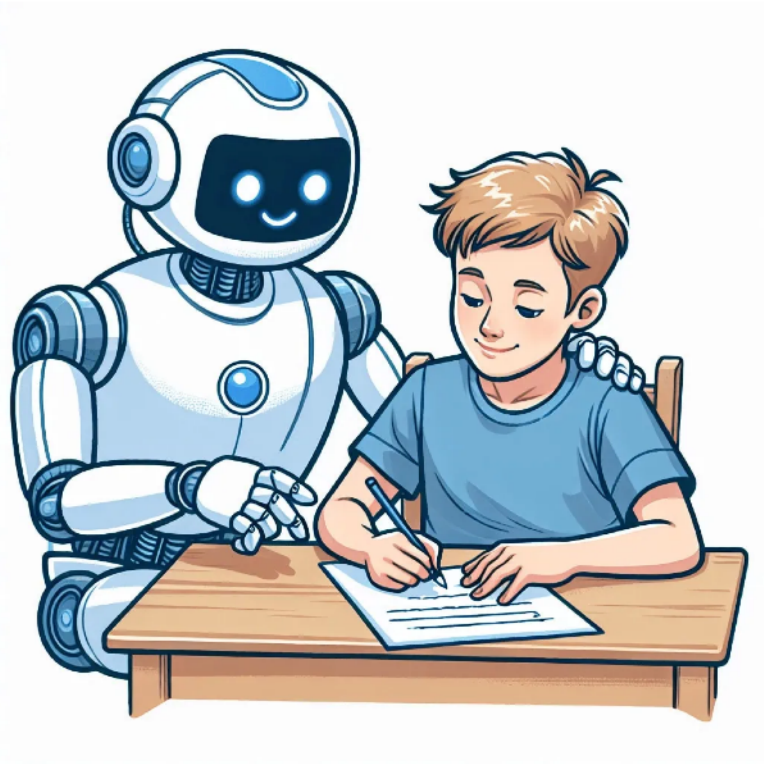
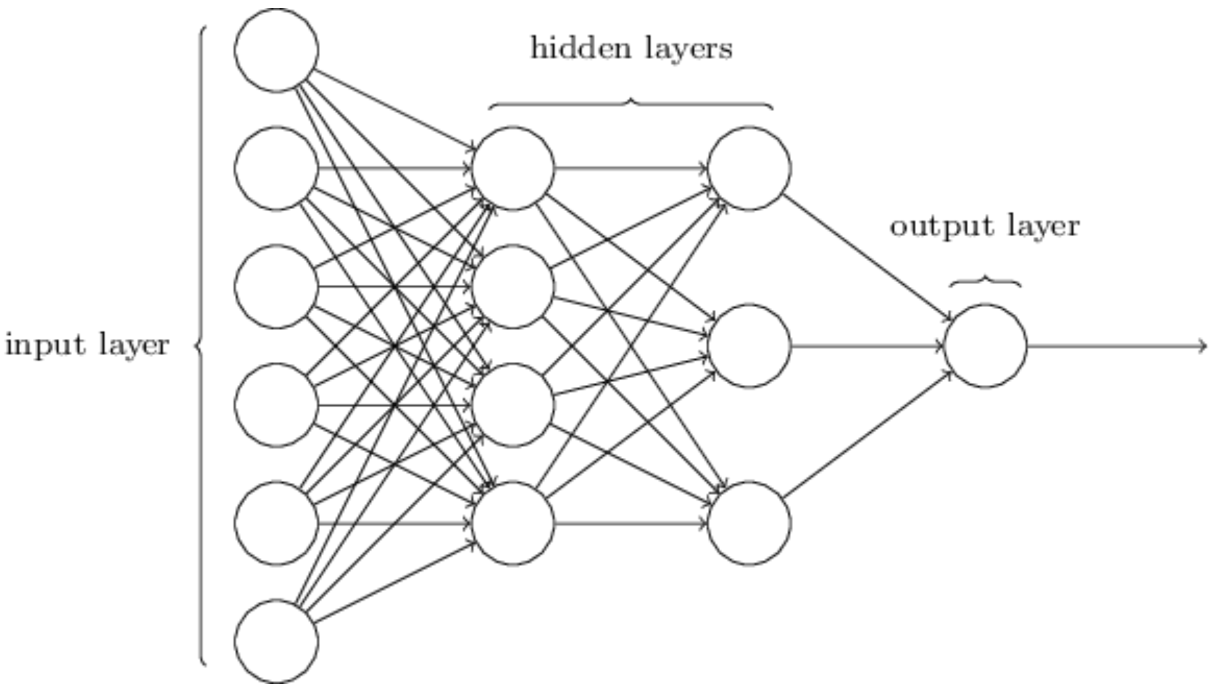
Content

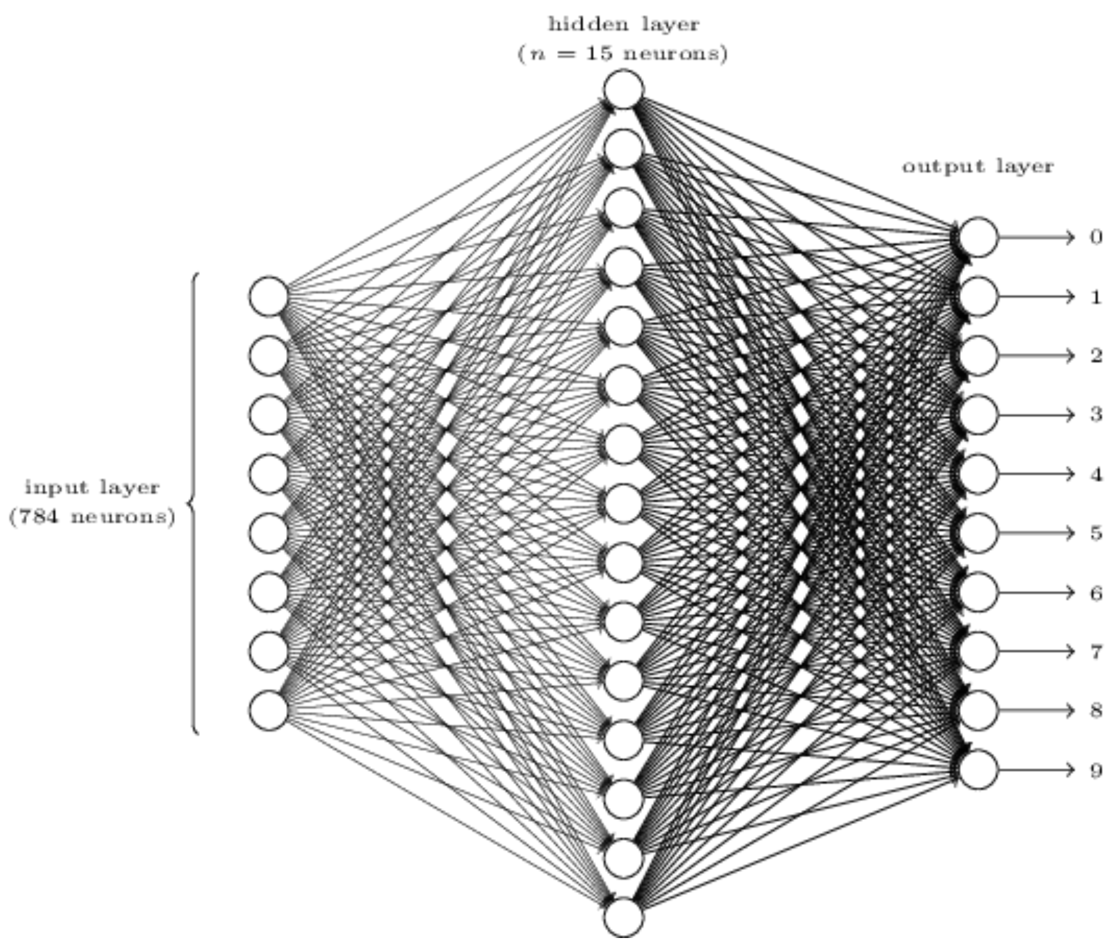


Content



Four-layer network with two hidden layers

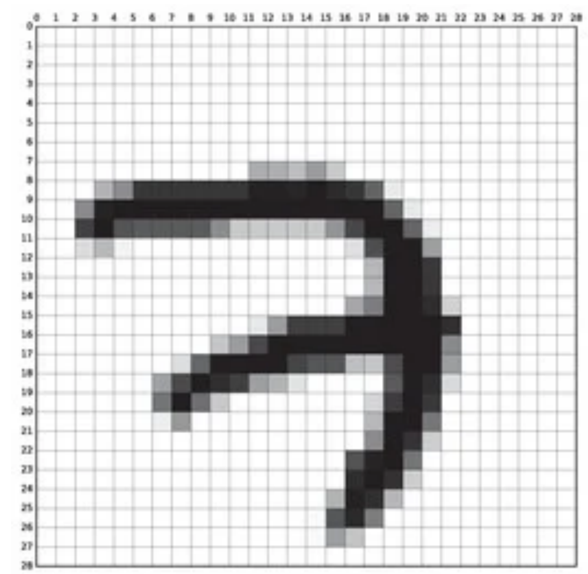




MNIST database of handwritten digits

Modified national institute of standards and technology database

60,000 training images and 10,000 testing images



(a) MNIST sample belonging to the digit '7'.

(b) 100 samples from the MNIST training set.

encode the intensities of the image pixels into the input neurons. If the image is a 28x28 greyscale image, then we'd have 28×28=784 input neurons, with the intensities scaled appropriately between 0 and 1.

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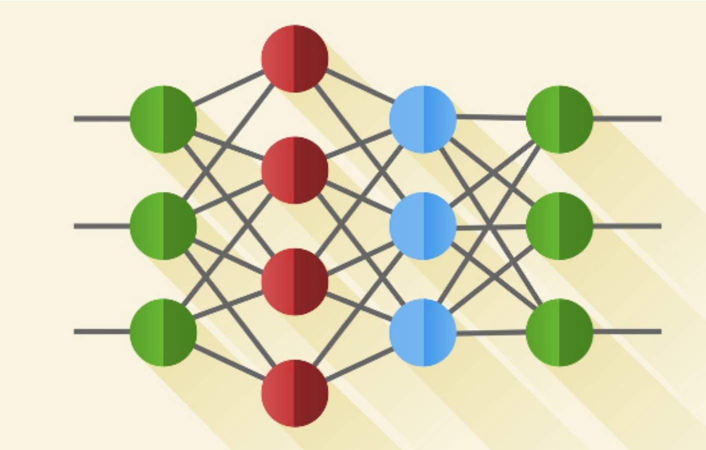
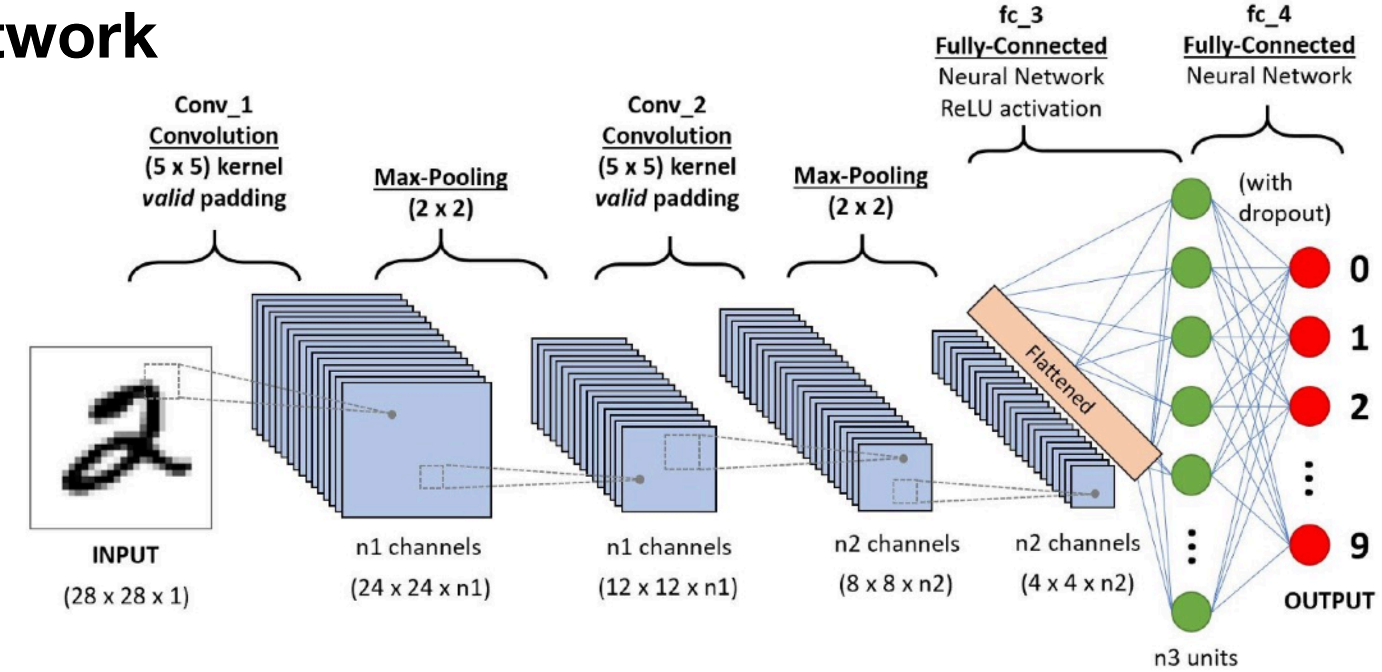
Connecting neurons to networks

Deep Learning: a network with 2 or more hidden layers with each layers containing large amount of neurons

Feed-forward network



Convolutional neural network

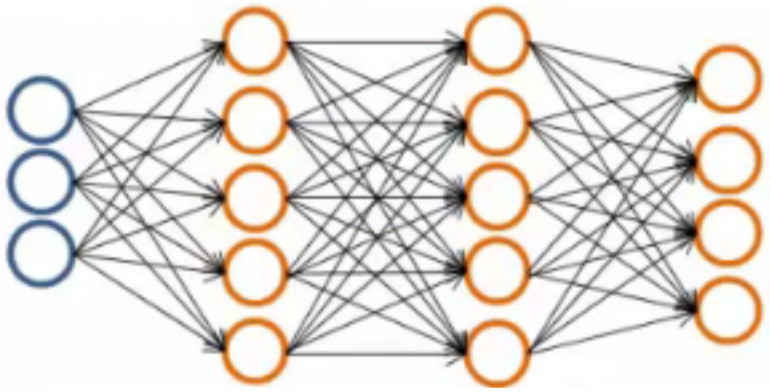


Generative neural network

Spiking neural network

https://towardsdatascience.com/introduction-to-math-behind-neural-networks-e8b60dbbdeba
https://towardsdatascience.com/nns-aynk-c34efe37f15a

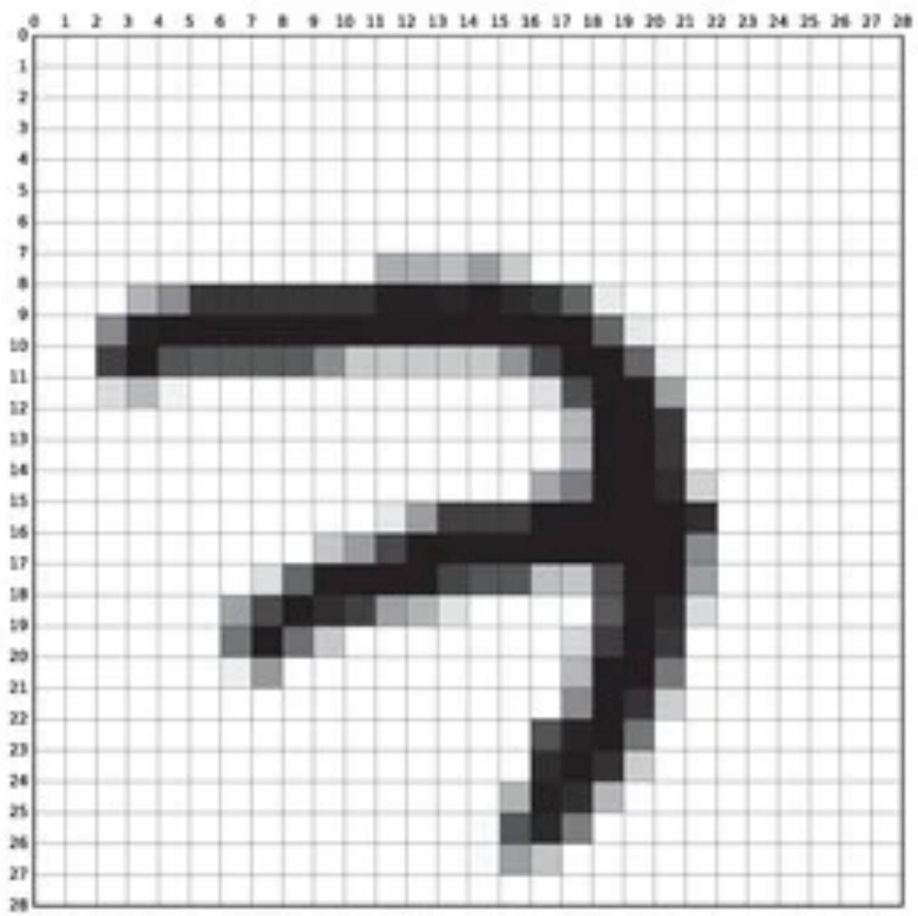
Neural network — Representation



Multiclass classification

$$h_\Theta(x) \in \mathbb{R}^4$$

Handwritten digital recognition problem - 10 possible categories (0-9)



(a) MNIST sample belonging to the digit '7'.

(b) 100 samples from the MNIST training set.

Mathematical representation for neutrons: Learning Algorithm

Backpropagation: backward propagation of errors

Computing the gradient of the cost function with respect to the weights

Cost function

$$J = \text{mean squared error} = \frac{1}{m}\sum_{i=1}^{m}(y_i - \hat{y}_i)^2$$

$$\frac{\partial J}{\partial w_i} = \frac{\partial J}{\partial \hat{y}} \times \frac{\partial \hat{y}}{\partial z} \times \frac{\partial z}{\partial w_i}$$

$$\frac{2}{m}\sum_{i=1}^{m}(y_i - \hat{y}_i) \qquad g(z)g(-z) \qquad x$$

$$1$$

$$\frac{\partial J}{\partial b} = \frac{\partial J}{\partial \hat{y}} \times \frac{\partial \hat{y}}{\partial z} \times \frac{\partial z}{\partial b}$$

$$z = w \cdot x + b = \theta^T x$$

$$y = \frac{1}{1 + \exp(-\theta^T x)}$$

$$g(z) = \frac{1}{1 + e^{-z}}$$

1. $g(z) + g(-z) = 1$
2. $g'(z) = g(z)(1 - g(z)) = g(z)g(-z)$
3. $g'(-z) = g'(z)$

Optimization: select best weights and bias for the perceptron

$$w_i = w_i - \alpha \frac{\partial J}{\partial w_i} \qquad b = b - \alpha \frac{\partial J}{\partial b}$$

Andrew Ng, Stanford University

http://www.holehouse.org/mlclass/

https://github.com/mxc19912008/Andrew-Ng-Machine-Learning-Notes

Michael Nielsen, scientist at home, the best reading material

http://neuralnetworksanddeeplearning.com

http://neuralnetworksanddeeplearning.com/chap1.html

http://neuralnetworksanddeeplearning.com/chap2.html

https://machinelearningmastery.com/implement-backpropagation-algorithm-scratch-python/

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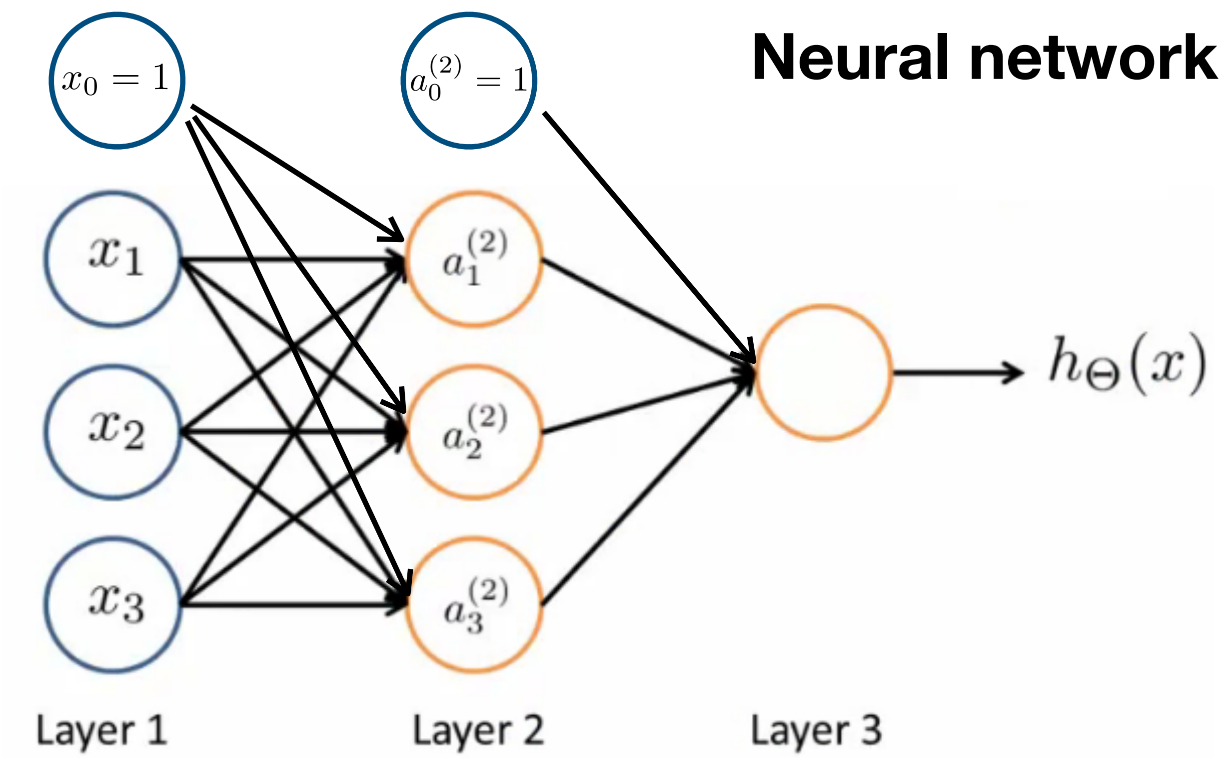
Early paper by
Li Wan, Matthew Zeiler, Sixin Zhang, Yann LeCun, Rob Fergus (2013)

http://yann.lecun.com/exdb/publis/pdf/wan-icml-13.pdf

Neural network — Representation



$a_i^{(l)}$ – activation of unit i in layer l

$$\Theta^{(l)} : [s^{(l+1)} \times (s^{(l)} + 1)]$$

the bias

Matrix of parameters mapping from layer (l) to layer (l+1)

Weights

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$a_1^{(2)} = g(\Theta_{10}^{(1)} x_0 + \Theta_{11}^{(1)} x_1 + \Theta_{12}^{(1)} x_2 + \Theta_{13}^{(1)} x_3)$$
$$a_2^{(2)} = g(\Theta_{20}^{(1)} x_0 + \Theta_{21}^{(1)} x_1 + \Theta_{22}^{(1)} x_2 + \Theta_{23}^{(1)} x_3) \qquad \Theta^{(1)} : [3 \times 4]$$
$$a_3^{(2)} = g(\Theta_{30}^{(1)} x_0 + \Theta_{31}^{(1)} x_1 + \Theta_{32}^{(1)} x_2 + \Theta_{33}^{(1)} x_3)$$
$$h_\Theta(x) = a_1^{(3)} = g(\Theta_{10}^{(2)} a_0^{(2)} + \Theta_{11}^{(2)} a_1^{(2)} + \Theta_{12}^{(2)} a_2^{(2)} + \Theta_{13}^{(2)} a_3^{(2)}) \qquad \Theta^{(2)} : [1 \times 4]$$

$$(\; z_1^2 = \theta_{10}^{(1)} x_0 + \theta_{11}^{(1)} x_1 + \theta_{12}^{(1)} x_2 + \theta_{13}^{1} x_3 \;, \; z_2^2, \; z_3^2 \;)$$

Vector representation

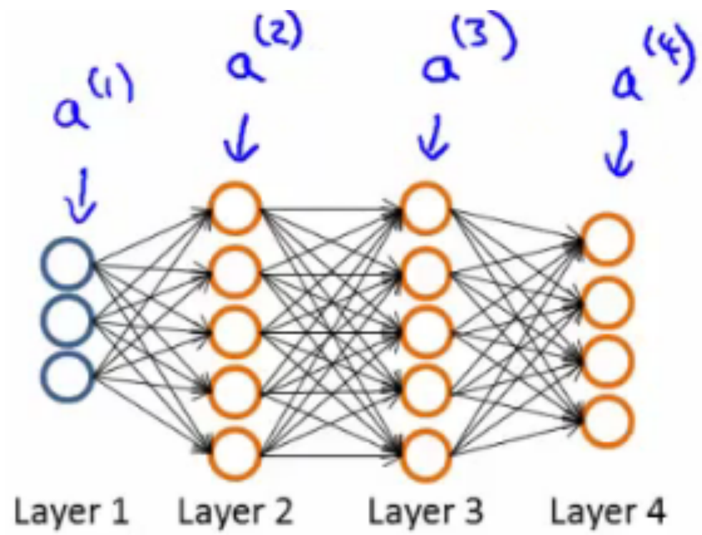
$$z^2 = \Theta^{(1)} x \qquad a^2 = g(z^2)$$

$$z^3 = \Theta^{(2)} a^2 \qquad h_\Theta(x) = a^3 = g(z^3)$$

Neural networks learns its own features

- Features in the hidden layer are calculated/learned - not original features
- Flexibility to learn whatever features it wants to feed into the final logistic regression calculation
- Hidden layers do the job, learn what gives the best final results to feed into final output layer
- Non-linear hypothesis

Neural network — Learning — back-propagation



$L = 4 \quad s^1 = 3 \quad s^2 = 5 \quad s^3 = 5 \quad s^4 = 4$

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Training set (in vector form)

$$\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \cdots, (x^m, y^m)\}$$

Forward propagation

$$a^1 = x$$
$$z^2 = \Theta^1 a^1 \ (\text{add } x_0)$$
$$a^2 = g(z^2)$$
$$z^3 = \Theta^2 a^2 \ (\text{add } a_0^2)$$
$$a^3 = g(z^3)$$
$$z^4 = \Theta^3 a^3 \ (\text{add } a_0^3)$$
$$a^4 = h_\Theta(x) = g(z^4)$$

Back propagation

δ_j^l – **the error of neuron j in layer l**

$$\delta^1 = (\Theta^1)^T \delta^2 - x$$
$$\delta^2 = (\Theta^2)^T \delta^3 .* g'(z^2)$$
$$\delta^3 = (\Theta^3)^T \delta^4 .* g'(z^3)$$
$$\delta^4 = (a^4 - y) .* g'(z^4)$$
$$\tilde{\delta}_j^4 = a_j^4 - y_j$$

from quadratic cost

$.*$ elementwise multiplication
Hadamard product

$$g'(z) = g(z) .* (1 - g(z))$$

$$\Theta^3 : [4 \times 5] \ ([4 \times 6] \text{ if include bias})$$
$$(\Theta^3)^T : [5 \times 4]$$
$$\delta^4 : [4 \times 1]$$
$$g'(z^3) : [5 \times 1]$$

Neural network — Learning — back-propagation

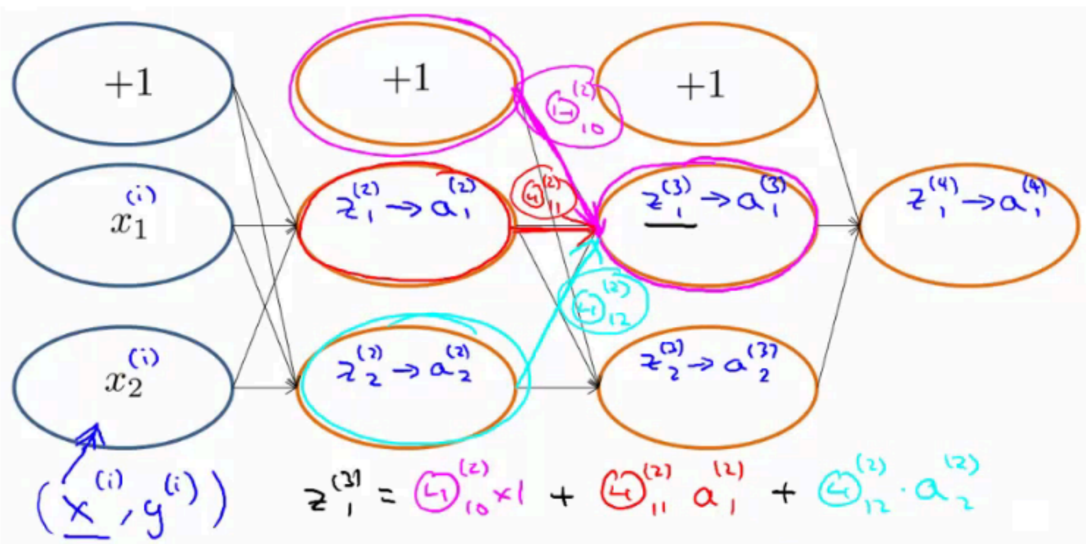
$$J(\Theta) = \frac{1}{2m}\sum_x ||y(x) - a^L(x)||^2 = \frac{1}{2m}\sum_i ||a_i^L - y_i||^2$$

Training set (in vector form) $\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \cdots, (x^m, y^m)\}$

$$\boxed{\frac{\partial J(\Theta)}{\partial \Theta_{ij}^{(l)}} = a_j^{(l)}\delta_i^{(l+1)} = a_{\textbf{in}}\delta_{\textbf{out}}}$$

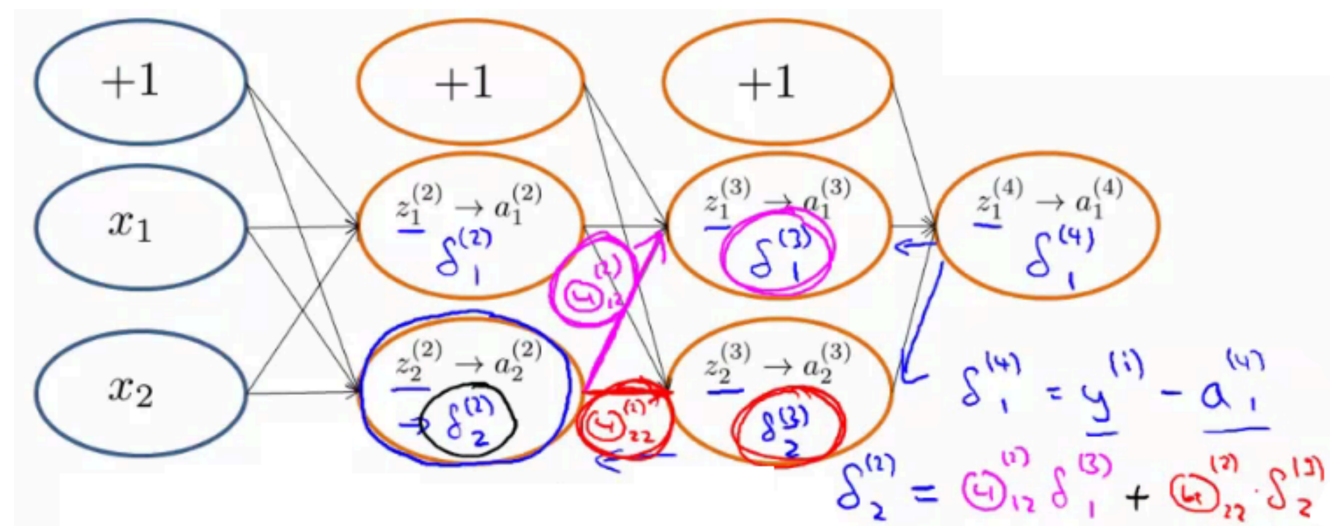
Loop through the training set $i = 1 \text{ to } m$

Forward propagation



Calculate activation values: weighted sum of the previous layer's activation values, weighted by the link parameters

Back propagation
How well is the network doing on example i ?



Calculate delta values: weighted sum of the next layer's delta values, weighted by the link parameters

$$\frac{\partial J}{\partial \Theta_{10}^{(1)}} = \delta_1^{(2)} \qquad \frac{\partial J}{\partial \Theta_{10}^{(2)}} = \delta_1^{(3)}$$

After executing the loop

$$\frac{\partial J}{\partial \Theta_{12}^{(1)}} = a_2^{(1)}\delta_1^{(2)} \qquad \frac{\partial J}{\partial \Theta_{12}^{(2)}} = a_2^{(2)}\delta_1^{(3)}$$

Update the cost function

$$\Theta_{ij}^{(l)} = \Theta_{ij}^{(l)} - \alpha\frac{\partial J(\Theta)}{\partial \Theta_{ij}^{(l)}}$$

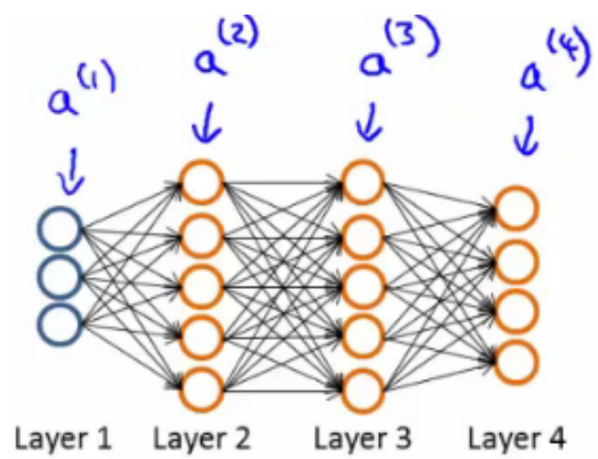
$$\frac{\partial J}{\partial \Theta_{22}^{(1)}} = a_2^{(1)}\delta_2^{(2)} \qquad \frac{\partial J}{\partial \Theta_{22}^{(2)}} = a_2^{(2)}\delta_2^{(3)}$$

Neural network — Learning put together

1) - pick a network architecture

Input units (x, a_1)
Output units (y)
Hidden units (Theta_1, a_2, Theta_2, a_3 …)

Normally more hidden units is better but more computationally expensive



2) - training a network

Randomly initiative the weights
Implement forward propagation to get a_L for any example x_i
Implement cost function J(Theta)
Implement back propagation to get partial derivatives

for i = 1:m {
    Forward propagation on (x^i, y^i) --> get activation (a) terms
    Back propagation on (x^i, y^i) --> get delta (δ) terms
    Compute $\Delta := \Delta^l + \delta^{l+1}(a^l)^T$
}

With this done compute the partial derivative terms

Use (gradient descent, CG or more advanced optimization) to minimise J(\Theta):
J(\Theta) is non-convex
Nonlinearity, local minimum, time, complexity …

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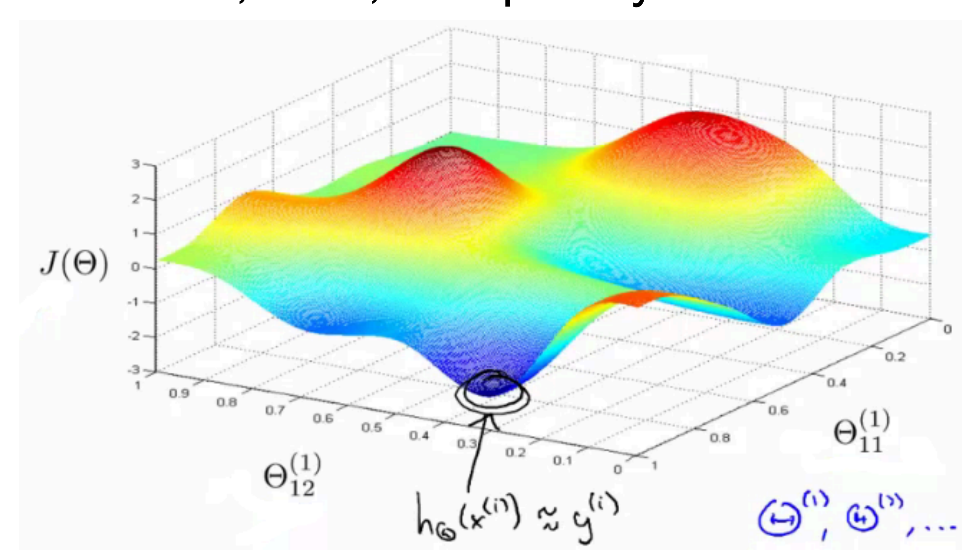
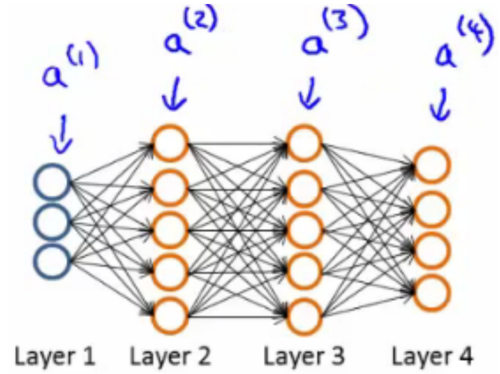




$$\delta_j^l = \frac{\partial J}{\partial z_j^l} \text{ measures the error of neuron } j \text{ in layer } l$$

An equation for the rate of change of the cost with respect to any weight in the network

Partial derivatives $$\frac{\partial}{\partial \Theta_{ij}^l} J(\Theta) = a_j^l \delta_i^{l+1}$$

Learn slowly if either the input neuron is low-activation, or if the output neuron has saturated, i.e., is either high- or low-activation.

Summary: the equations of backpropagation

$$\delta^L = \nabla_a C \odot \sigma'(z^L) \qquad \text{(BP1)}$$

$$\delta^l = ((w^{l+1})^T \delta^{l+1}) \odot \sigma'(z^l) \qquad \text{(BP2)}$$

$$\frac{\partial C}{\partial b_j^l} = \delta_j^l \qquad \text{(BP3)}$$

$$\frac{\partial C}{\partial w_{jk}^l} = a_k^{l-1} \delta_j^l \qquad \text{(BP4)}$$

Prove this $$\delta_j^2 = \frac{\partial J}{\partial z_j^2} = \sum_k \frac{\partial J}{\partial z_k^3} \frac{\partial z_k^3}{\partial z_j^2} = \sum_k \delta_k^3 \frac{\partial z_k^3}{\partial z_j^2}$$

$$z_k^3 = \sum_j \Theta_{kj}^2 a_j^2 = \sum_j \Theta_{kj}^2 g(z_j^2) \qquad \frac{\partial z_k^3}{\partial z_j^2} = \Theta_{kj}^2 g'(z_j^2)$$

$$\delta_j^2 = \sum_k \Theta_{kj}^2 \, \delta_k^3 \, g'(z_j^2) \qquad \delta^2 = (\Theta^2)^T \delta^3 .* g'(z^2)$$

Notice the shift of l by 1, index jungle

Prove this $$\frac{\partial J}{\partial \Theta^2} = \frac{\partial J}{\partial z^3} \frac{\partial z^3}{\partial \Theta^2} = \delta^3 a^2 = a^2 \delta^3$$

$$\delta^3 = \frac{\partial J}{\partial z^3} \qquad z^3 = \Theta^2 a^2$$

Michael Nielsen http://neuralnetworksanddeeplearning.com/chap2.html

In particular, given a mini-batch of m training examples, the following algorithm applies a gradient descent learning step based on that mini-batch

1. **Input a set of training examples**

2. **For each training example x:** Set the corresponding input activation $a^{x,1}$, and perform the following steps:

    - **Feedforward:** For each $l = 2, 3, \ldots, L$ compute $z^{x,l} = w^l a^{x,l-1} + b^l$ and $a^{x,l} = \sigma(z^{x,l})$.

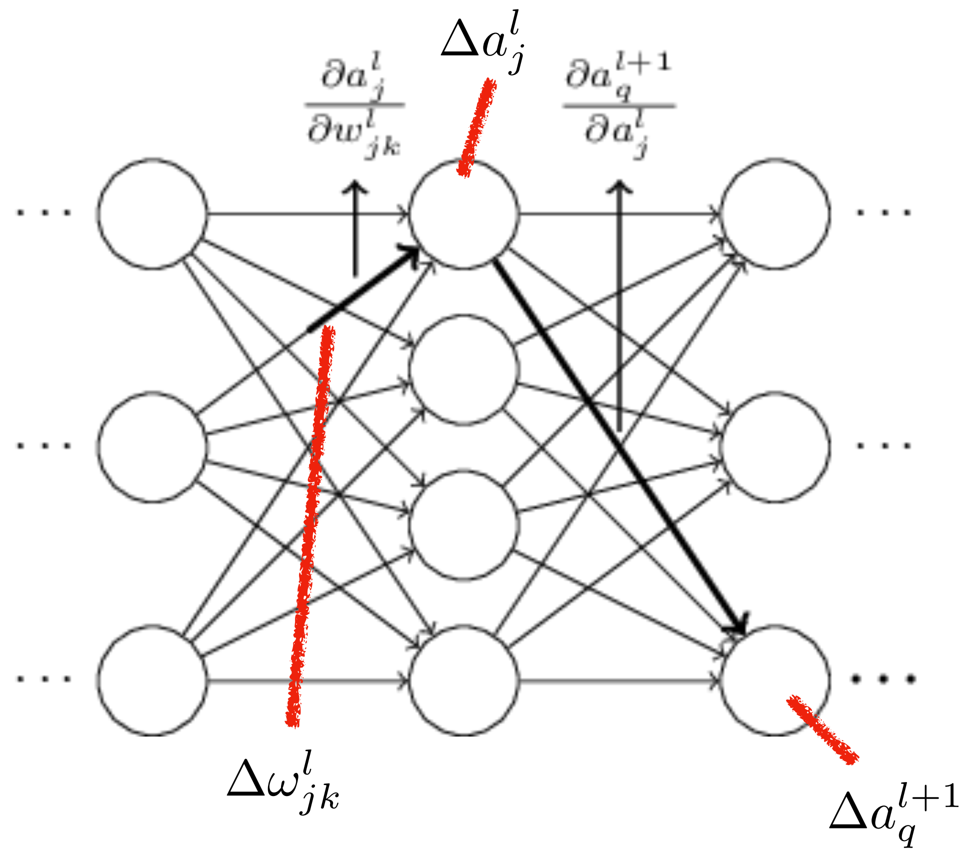
    - **Output error $\delta^{x,L}$:** Compute the vector $\delta^{x,L} = \nabla_a C_x \odot \sigma'(z^{x,L})$.

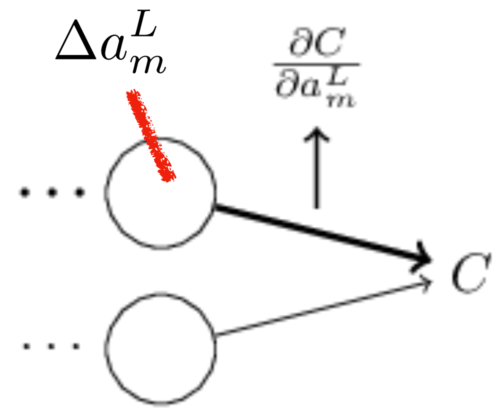
    - **Backpropagate the error:** For each $l = L-1, L-2, \ldots, 2$ compute $\delta^{x,l} = ((w^{l+1})^T \delta^{x,l+1}) \odot \sigma'(z^{x,l})$.

3. **Gradient descent:** For each $l = L, L-1, \ldots, 2$ update the weights according to the rule $w^l \rightarrow w^l - \frac{\eta}{m} \sum_x \delta^{x,l} (a^{x,l-1})^T$, and the biases according to the rule $b^l \rightarrow b^l - \frac{\eta}{m} \sum_x \delta^{x,l}$.

Michael Nielsen http://neuralnetworksanddeeplearning.com/chap2.html

Sigmoid is differentiable





$$\Delta C \approx \frac{\partial C}{\partial w_{jk}^l} \Delta w_{jk}^l.$$

$$\Delta a_j^l \approx \frac{\partial a_j^l}{\partial w_{jk}^l} \Delta w_{jk}^l \qquad \Delta a_q^{l+1} \approx \frac{\partial a_q^{l+1}}{\partial a_j^l} \frac{\partial a_j^l}{\partial w_{jk}^l} \Delta w_{jk}^l \qquad \cdots \qquad \Delta C \approx \frac{\partial C}{\partial a_m^L} \frac{\partial a_m^L}{\partial a_n^{L-1}} \frac{\partial a_n^{L-1}}{\partial a_p^{L-2}} \cdots \frac{\partial a_q^{l+1}}{\partial a_j^l} \frac{\partial a_j^l}{\partial w_{jk}^l} \Delta w_{jk}^l$$

$$\Delta C \approx \sum_{mnp\ldots q} \frac{\partial C}{\partial a_m^L} \frac{\partial a_m^L}{\partial a_n^{L-1}} \frac{\partial a_n^{L-1}}{\partial a_p^{L-2}} \cdots \frac{\partial a_q^{l+1}}{\partial a_j^l} \frac{\partial a_j^l}{\partial w_{jk}^l} \Delta w_{jk}^l$$

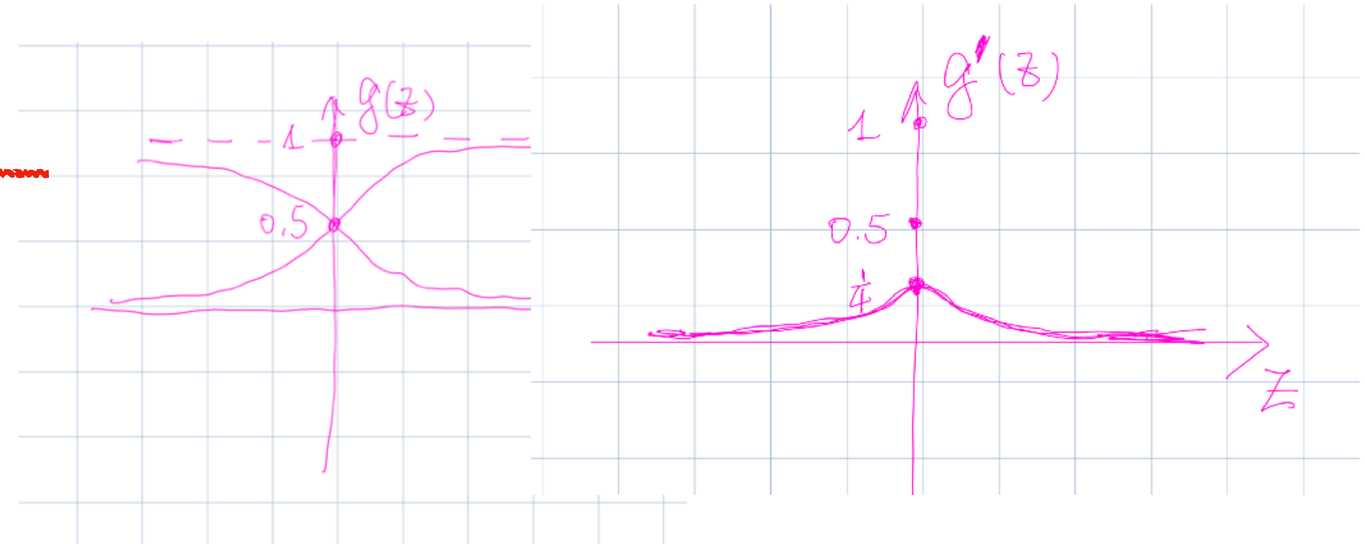
$$\frac{\partial C}{\partial w_{jk}^l} = \sum_{mnp\ldots q} \frac{\partial C}{\partial a_m^L} \frac{\partial a_m^L}{\partial a_n^{L-1}} \frac{\partial a_n^{L-1}}{\partial a_p^{L-2}} \cdots \frac{\partial a_q^{l+1}}{\partial a_j^l} \frac{\partial a_j^l}{\partial w_{jk}^l}$$

http://neuralnetworksanddeeplearning.com/chap2.html

Improving the way neural networks learn

$$\frac{\partial}{\partial \Theta_{ij}^{l}} J(\Theta) = a_j^l \delta_i^{l+1} \qquad \delta^3 = (\Theta^3)^T \delta^4 .* g'(z^3)$$

Learning slowdown when output neuron saturates



Cross-entropy as cost function (cost function of logistic regression)

$$J(\Theta) = -\frac{1}{m} \sum_{i=1}^{m} [y_i \ln(a_i^L) + (1 - y_i) \ln(1 - a_i^L)]$$

$$\frac{\partial J}{\partial \theta_i} = \frac{1}{m} \sum_{i=1}^{m} x_i (g(z_i) - y_i)$$

Error in output control the learning

$$\frac{\partial J}{\partial b} = \frac{1}{m} \sum_{i=1}^{m} (g(z_i) - y_i)$$

Regularization

$$J(\Theta) = -\frac{1}{m} \sum_{i=1}^{m} [y_i \ln(a_i^L) + (1 - y_i) \ln(1 - a_i^L)] + \frac{\lambda}{2m} \sum_{\Theta} \Theta^2$$

http://neuralnetworksanddeeplearning.com/chap3.html