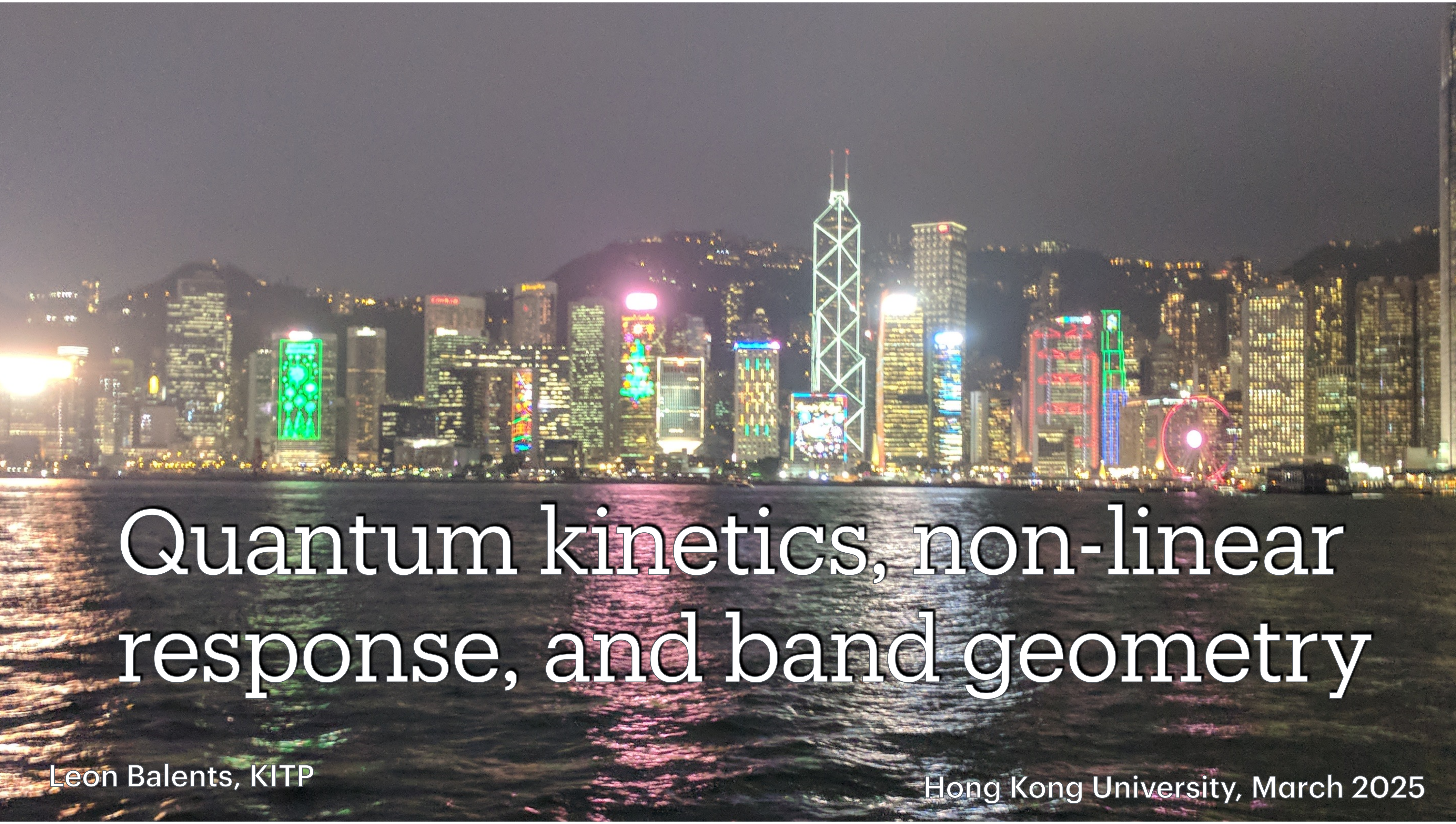


Quantum kinetics, non-linear response, and band geometry

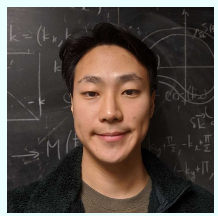
Leon Balents, KITP

Hong Kong University, March 2025

Collaborators

Lucile Savary
FACTS

Léo Mangeolle
TUM



Takamori Park
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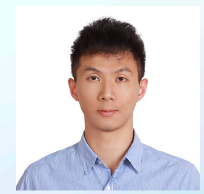
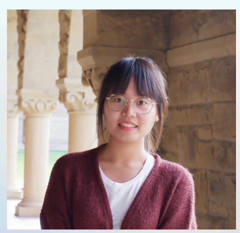
Xiaoyang Huang
CU Boulder

Kaixiang Su
UCSB



Ren-Bo Wang
FACTS

Prequel: PRB 2024



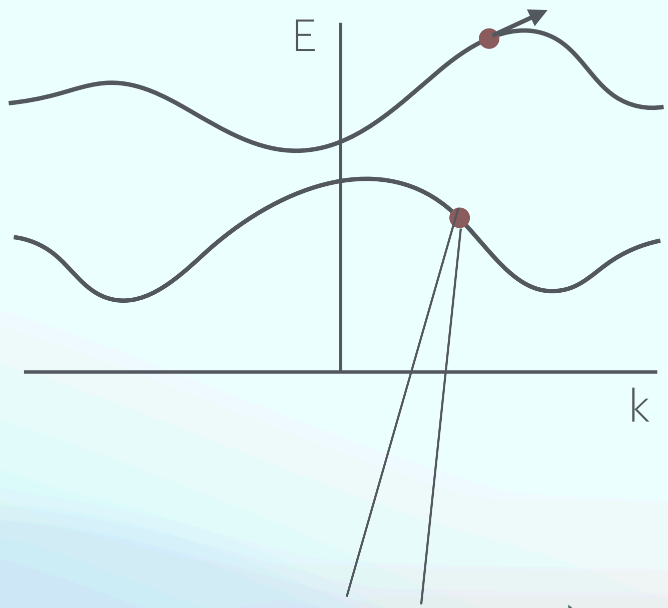
Wen Wang
UCSB

Jia-Xin Zhang
FACTS

Outline

- Bands and quantum geometry
- Quantum geometric nesting in flat bands
- Doing semiclassics correctly in dispersive bands with quantum geometry

Bands



$\epsilon_n(k)$ → $\mathbf{v}_n(k)$

Energetics

$|\psi_{nk}\rangle$ →

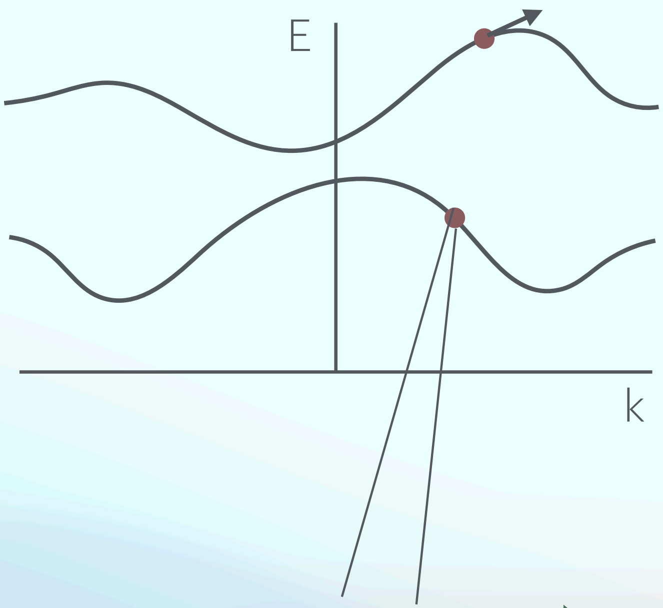
Geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Quantum geometric nesting

Bands



$|\psi_{nk}\rangle$ → Geometry

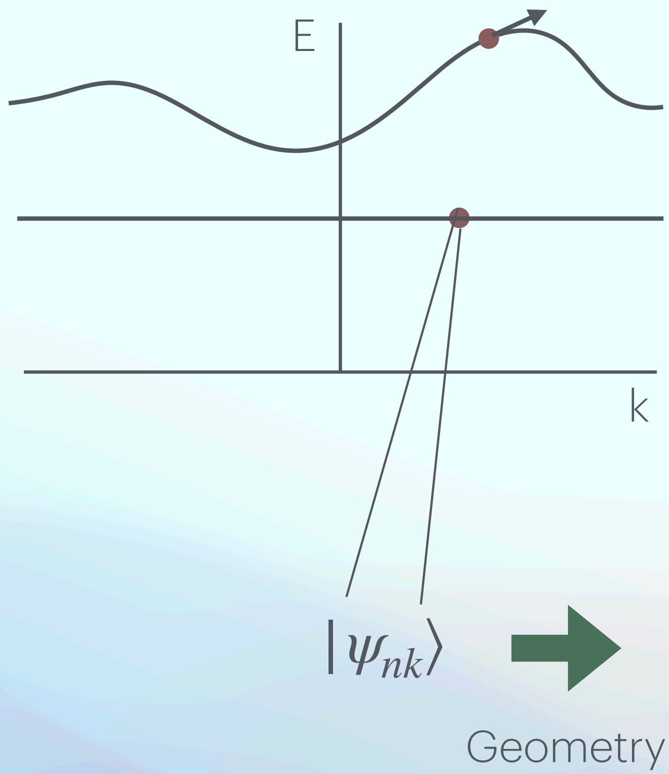
$\epsilon_n(k)$ → $\mathbf{v}_n(k)$

Energetics

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Flat bands

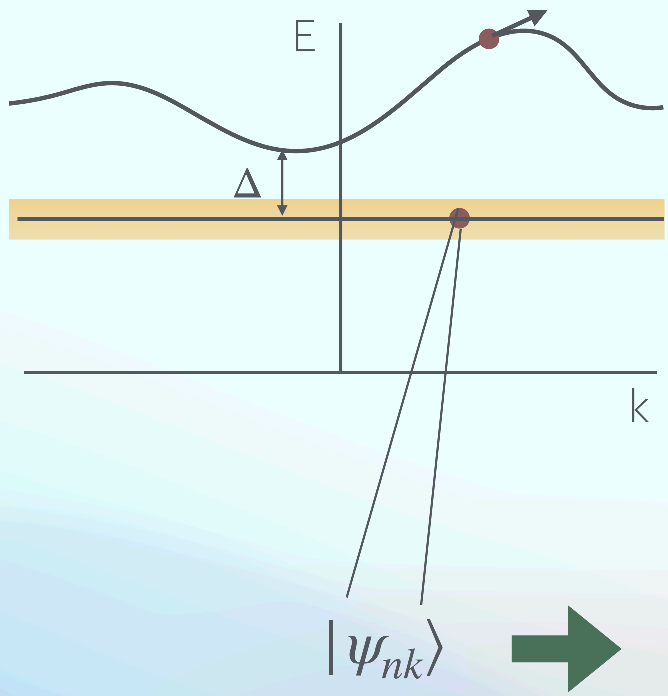


In the flat band limit, there is *only* geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Flat bands



Geometry

In the flat band limit, there is *only* geometry

Interaction $U \ll \Delta$

How does geometry determine the instabilities?

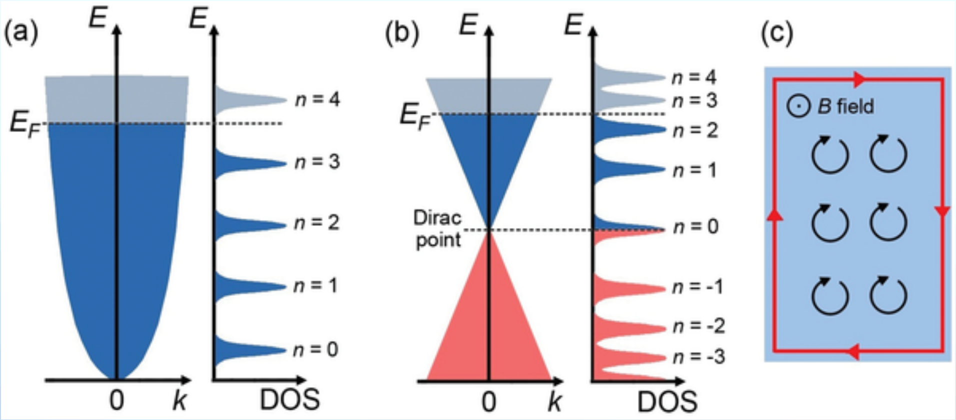
$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

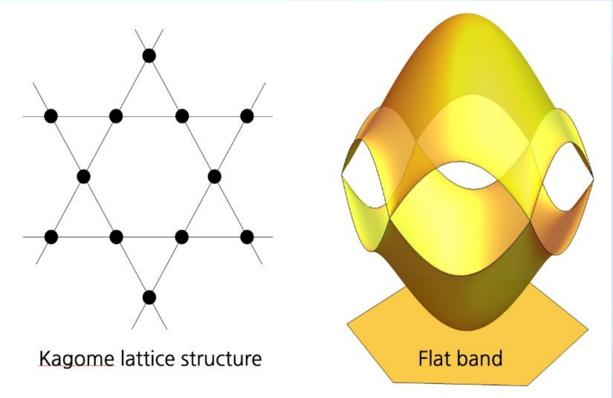
Flat bands

Generally a result of interference in electron motion, which can be induced by frustration or by magnetic flux.

Landau levels

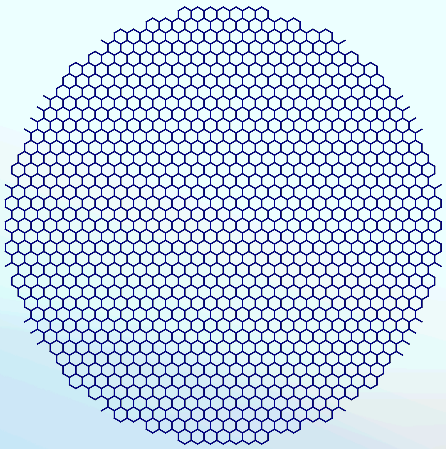


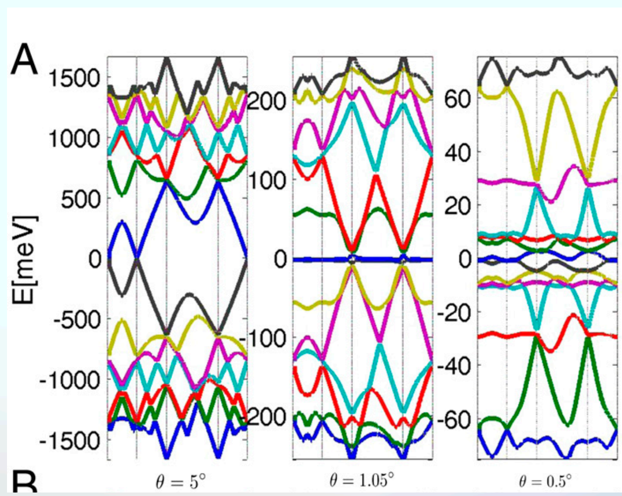
Frustrated hopping



Flat bands

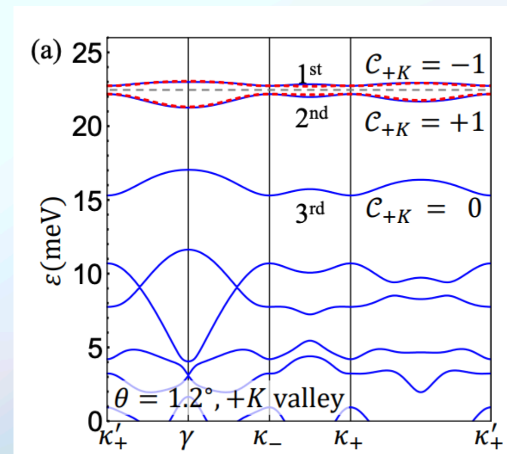
moiré





Bistritzer+Macdonald, 2011

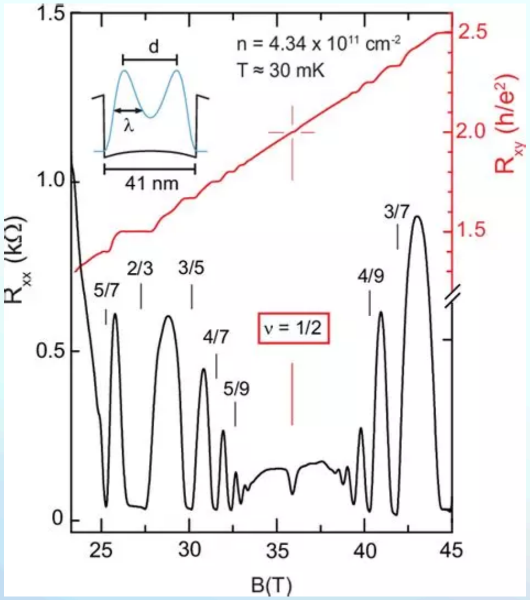
Graphene



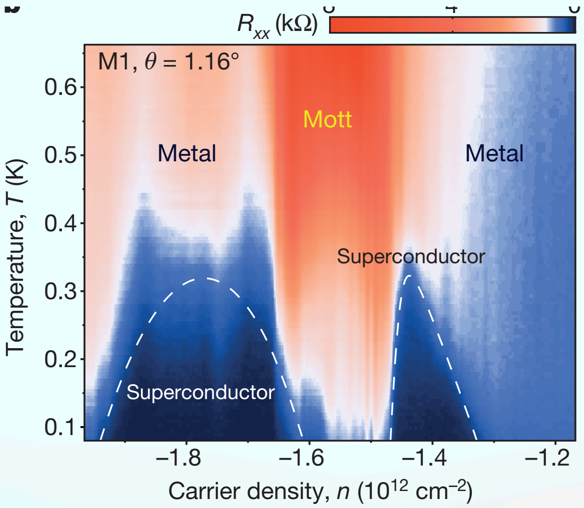
F. Wu *et al*, 2019

TMDs

Correlation physics

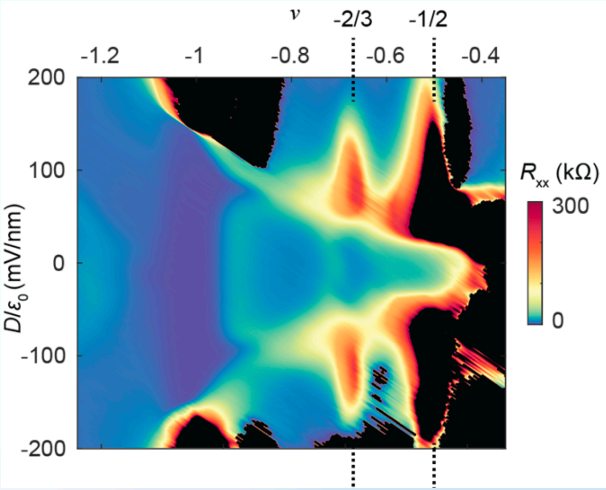


FQHE



Cao *et al*, 2018

Superconductivity and correlated insulators in TBG



H. Park *et al*, 2023

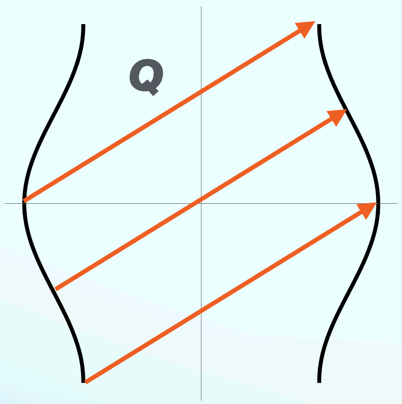
FQAHE in twisted MoTe2

So many different orders

- Ferromagnetism
- Antiferromagnetism
- Intervalley coherence
- Fractional quantum Hall effect
- Stripes
- Charge density wave
- Loop currents
- Superconductivity (s-wave, p-wave, d-wave, FFLO...)

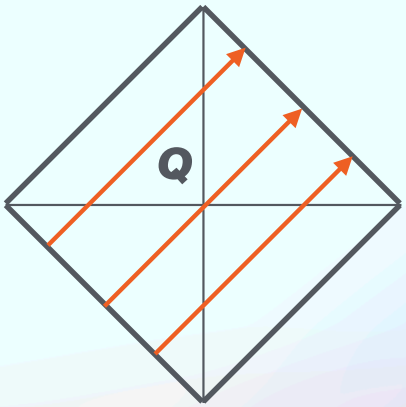
Nesting

Within a *dispersive* band, Fermi surface structure controls instabilities



Quasi-1d

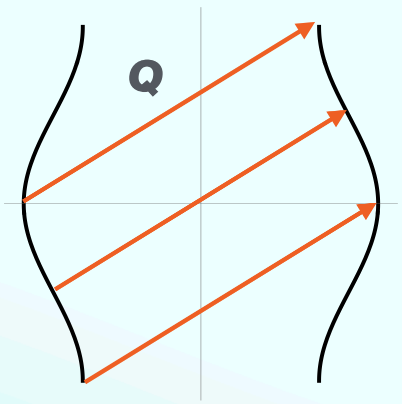
"$2k_F$" CDW/SDW



Square lattice Hubbard

(π, π) antiferromagnet

Susceptibility



Restricts to near Fermi surface

$$\chi(Q,\omega) \sim \sum_k \frac{n_F(\epsilon_{k+Q}) - n_F(\epsilon_k)}{\omega - \epsilon_k + \epsilon_{k+Q}}$$

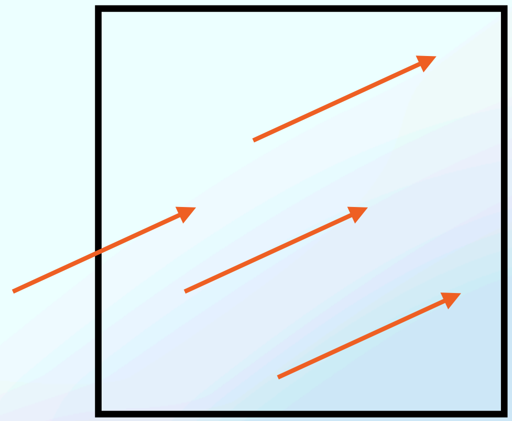
Enhanced susceptibility when this vanishes for extended range of k

What happens when the band is flat?? No Fermi surface!

Quantum geometric nesting

Z. Han *et al*, 2024

For flat band (or bands) at the Fermi energy, susceptibility is rendered finite only by thermal fluctuations. But there is still structure.



$$\chi_{\boldsymbol{Q}} = \frac{1}{T}\sum_{\boldsymbol{k}} \mathrm{Tr}\left[\mathcal{O}^{\dagger}(\boldsymbol{k})P(\boldsymbol{k}+\boldsymbol{Q})\mathcal{O}(\boldsymbol{k})P(\boldsymbol{k})\right].$$

Order parameter

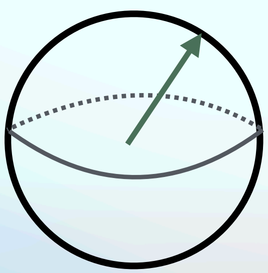
Projector

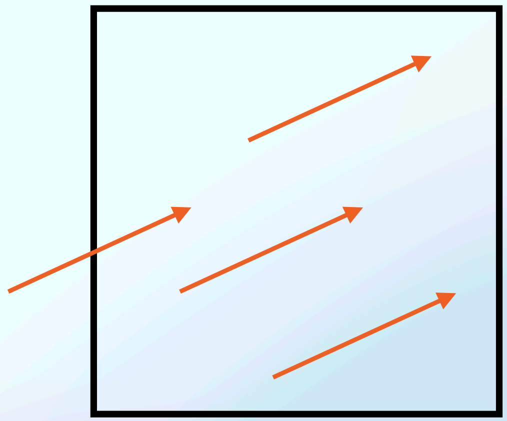
$$P(\boldsymbol{k}) = \sum_{n\in\mathrm{FB}} |\psi_{n\boldsymbol{k}}\rangle\langle\psi_{n\boldsymbol{k}}|$$

Quantum geometric nesting

Bloch vector representation

e.g. 2-band $$P(\boldsymbol{k}) = \frac{1}{2} + \frac{1}{2}\boldsymbol{b}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$$





$$\mathcal{O}(\boldsymbol{k}) = o(\boldsymbol{k}) + \boldsymbol{o}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$$

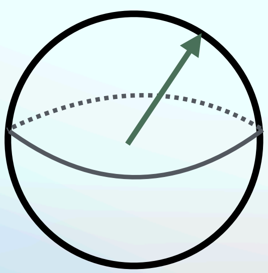
Quantum geometric nesting

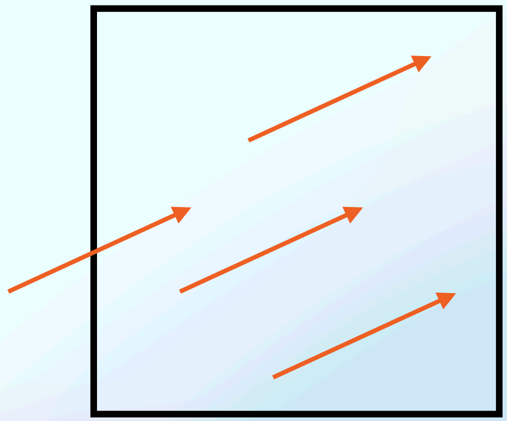
Bloch vector representation

N bands, N_L flat

$$P(\boldsymbol{k}) = \frac{N_L}{N} + \frac{1}{2}\boldsymbol{b}(\boldsymbol{k}) \cdot \boldsymbol{\lambda}$$

$\boldsymbol{\lambda} \in \mathrm{SU(N)}$



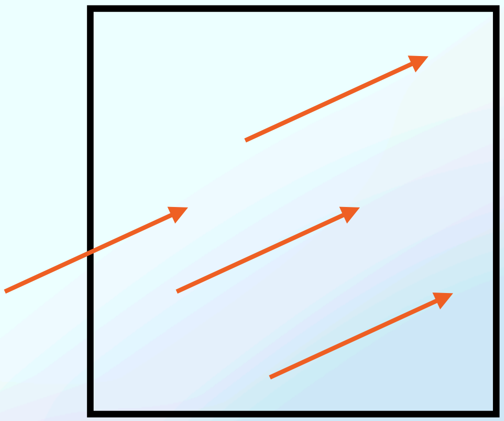


$$\mathcal{O}(\boldsymbol{k}) = o(\boldsymbol{k}) + \boldsymbol{o}(\boldsymbol{k}) \cdot \boldsymbol{\lambda}$$

Quantum geometric nesting

$$\chi(\boldsymbol{Q}) = \sum_{\boldsymbol{k}} \frac{\mathrm{Tr}[\mathcal{O}^\dagger(\boldsymbol{k})\mathcal{O}(\boldsymbol{k})]}{T} \left(\frac{N_L^2}{N} + \frac{1}{2} \tilde{\boldsymbol{b}}_{\boldsymbol{o}}(\boldsymbol{k}+\boldsymbol{Q}) \cdot \boldsymbol{b}(\boldsymbol{k}) \right),$$

Relates quantum geometry at **k** and **k+Q**



$$\tilde{\boldsymbol{b}}_{\boldsymbol{o}}(\boldsymbol{k}+\boldsymbol{Q}) \equiv \boldsymbol{b}(\boldsymbol{k}+\boldsymbol{Q}) - N\left[\hat{\boldsymbol{o}}(\boldsymbol{k}) \times \boldsymbol{b}(\boldsymbol{k}+\boldsymbol{Q}) \times \hat{\boldsymbol{o}}(\boldsymbol{k})\right]$$

QGN condition:

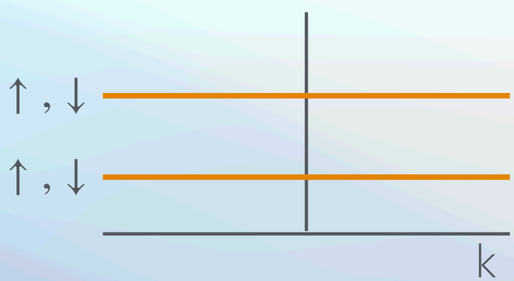
$$\tilde{\boldsymbol{b}}_{\boldsymbol{o}}(\boldsymbol{k}+\boldsymbol{Q}) \parallel \boldsymbol{b}(\boldsymbol{k}), \qquad \forall \boldsymbol{k} \in \mathrm{BZ}$$

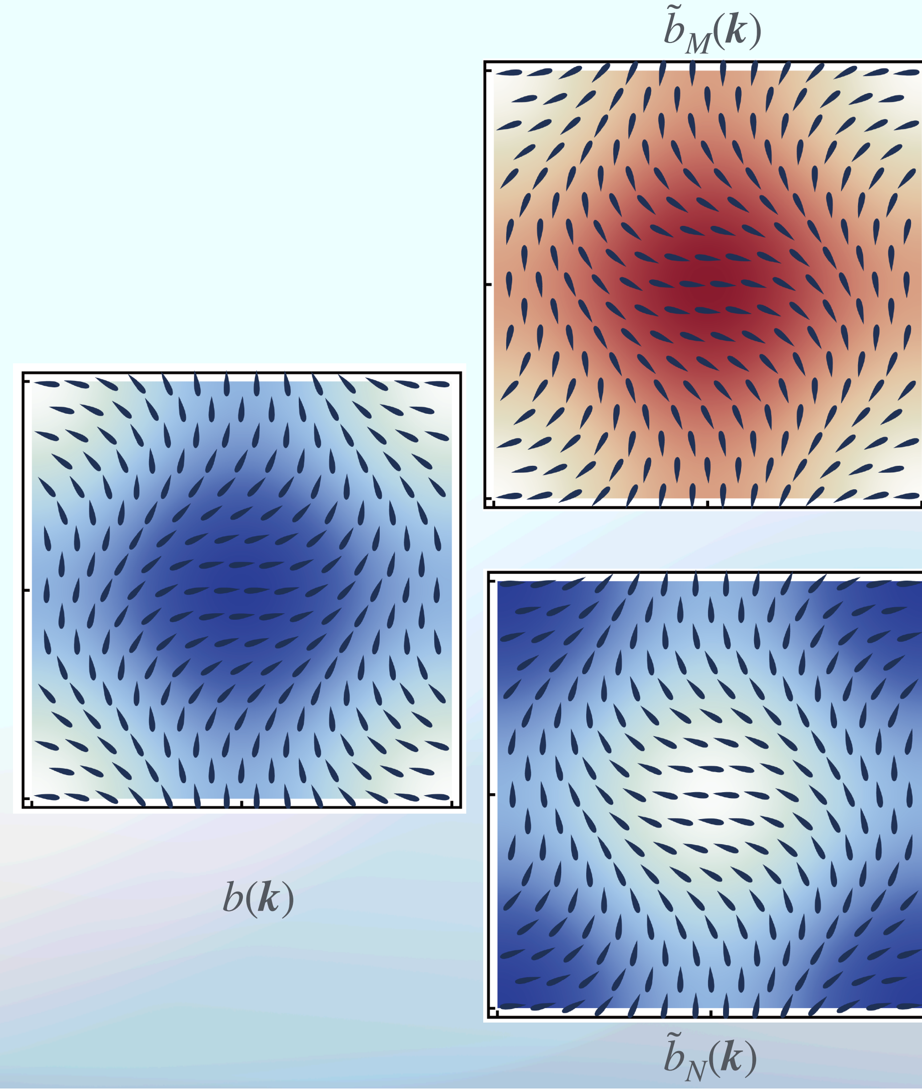
Example

JS Hofmann *et al*, 2022,2023

$$H_0 = t\sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} \begin{pmatrix} -\mu & -ie^{i\alpha^\uparrow_k} & 0 & 0 \\ ie^{-i\alpha^\uparrow_k} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha^\downarrow_k} \\ 0 & 0 & -ie^{i\alpha^\downarrow_k} & -\mu \end{pmatrix} c_{\boldsymbol{k}}$$

$$\alpha^\sigma_k = \eta_\sigma\left(\cos k_x + \cos k_y\right)$$



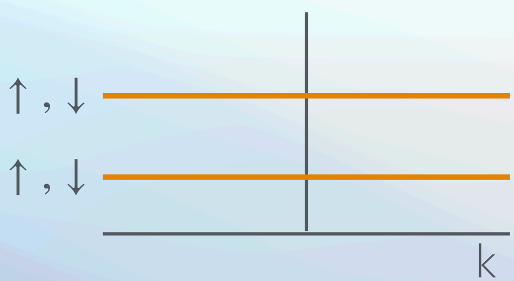


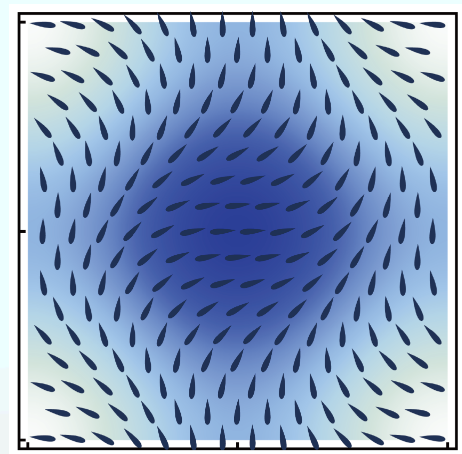
Example

JS Hofmann *et al*, 2022,2023

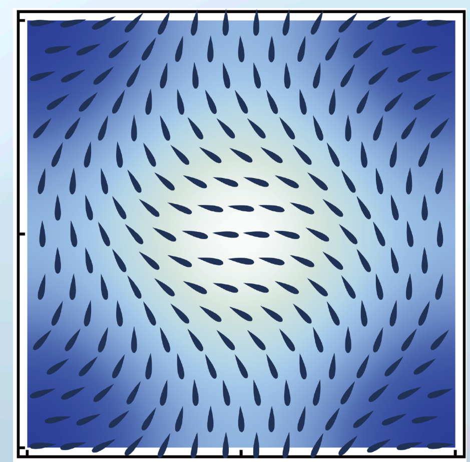
$$H_0 = t \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} \begin{pmatrix} -\mu & -ie^{i\alpha_k^\uparrow} & 0 & 0 \\ ie^{-i\alpha_k^\uparrow} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_k^\downarrow} \\ 0 & 0 & -ie^{i\alpha_k^\downarrow} & -\mu \end{pmatrix} c_{\boldsymbol{k}}$$

$$\alpha_k^\sigma = \eta_\sigma \left(\cos k_x + \cos k_y\right)$$





$b(\boldsymbol{k})$



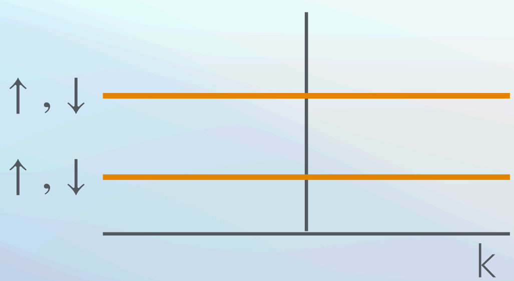
$\tilde{b}_N(\boldsymbol{k})$

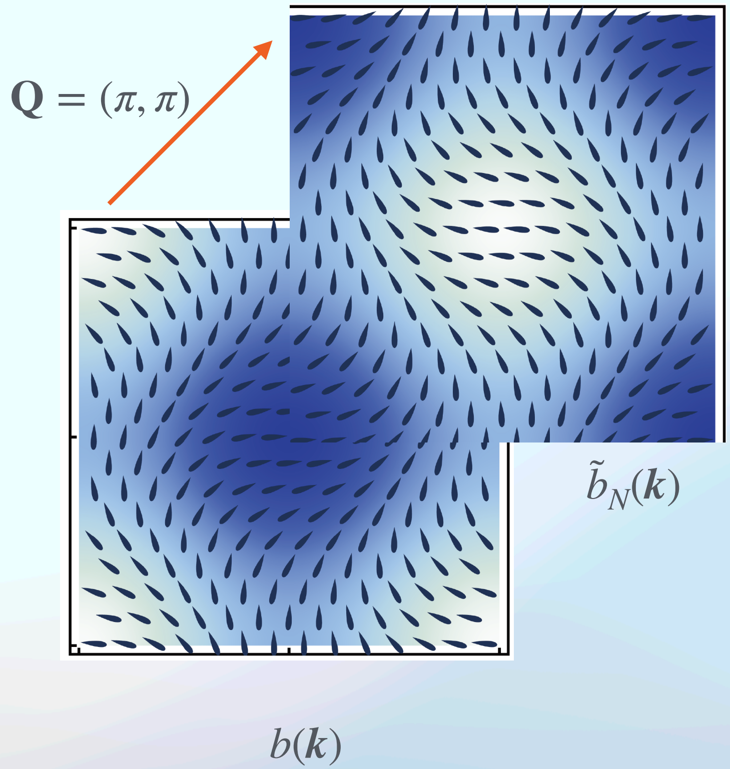
Example

JS Hofmann *et al*, 2022,2023

$$H_0 = t \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} \begin{pmatrix} -\mu & -ie^{i\alpha_k^\uparrow} & 0 & 0 \\ ie^{-i\alpha_k^\uparrow} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_k^\downarrow} \\ 0 & 0 & -ie^{i\alpha_k^\downarrow} & -\mu \end{pmatrix} c_{\boldsymbol{k}}$$

$$\alpha_k^\sigma = \eta_\sigma \left(\cos k_x + \cos k_y\right)$$





QGN!

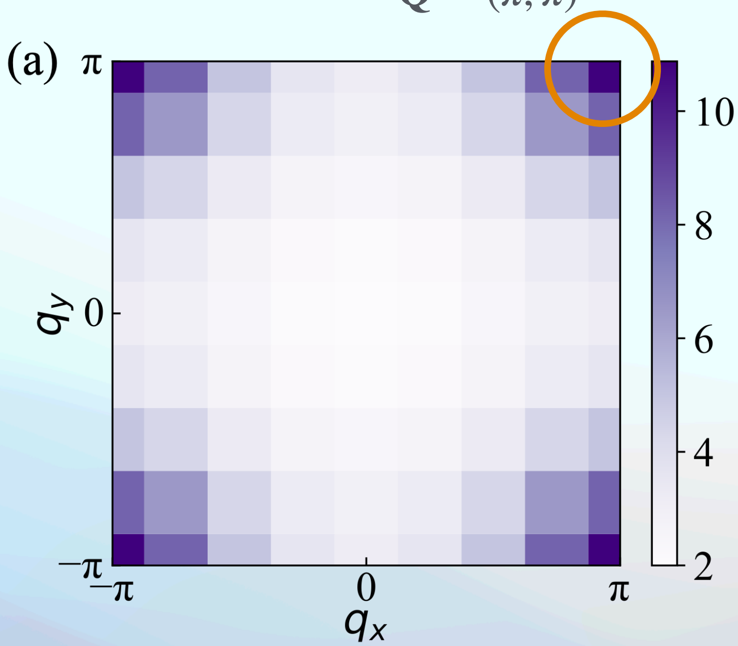
QMC Simulation

Include Hubbard interaction $H = H_0 + U \sum_i \left(n_{i,A}^2 + n_{i,B}^2 \right)$

Use dQMC for 8x8 lattice

$\mathbf{Q} = (\pi, \pi)$

U/t=1



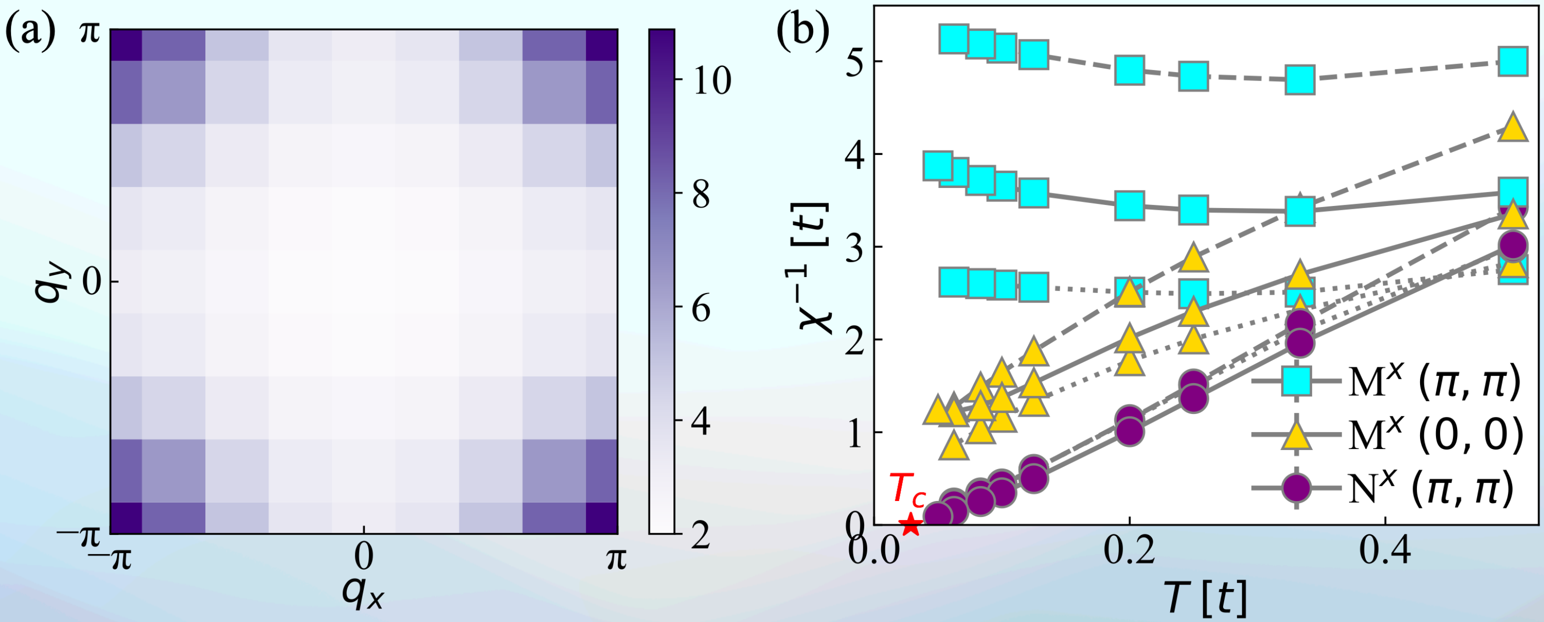
T/t = 0.05

QMC Simulation

Include Hubbard interaction $H = H_0 + U \sum_i \left(n_{i,A}^2 + n_{i,B}^2 \right)$

Use dQMC for 8x8 lattice

U/t=1



T/t = 0.05

Superconductivity

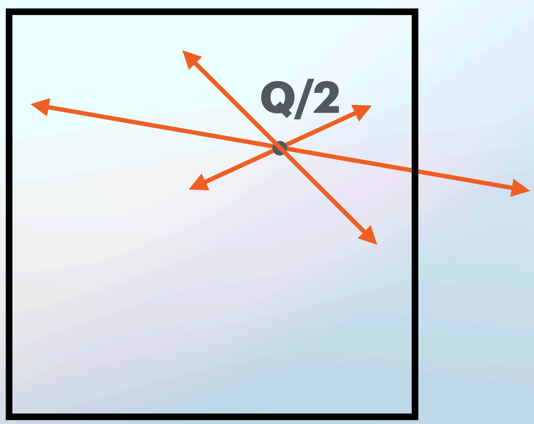
With attractive interactions in the flat band, can get exotic superconductivity

$$\chi_{\boldsymbol{Q}}^{\mathrm{pp}} = \frac{1}{4T}\frac{|\nu|}{\operatorname{arctanh}|\nu|}\sum_{\boldsymbol{k}} o_0^2(\boldsymbol{k})\left(Nn^2 + \frac{1}{2}\boldsymbol{b}^R(\boldsymbol{k}+\boldsymbol{Q})\cdot\boldsymbol{b}(-\boldsymbol{k})\right)$$

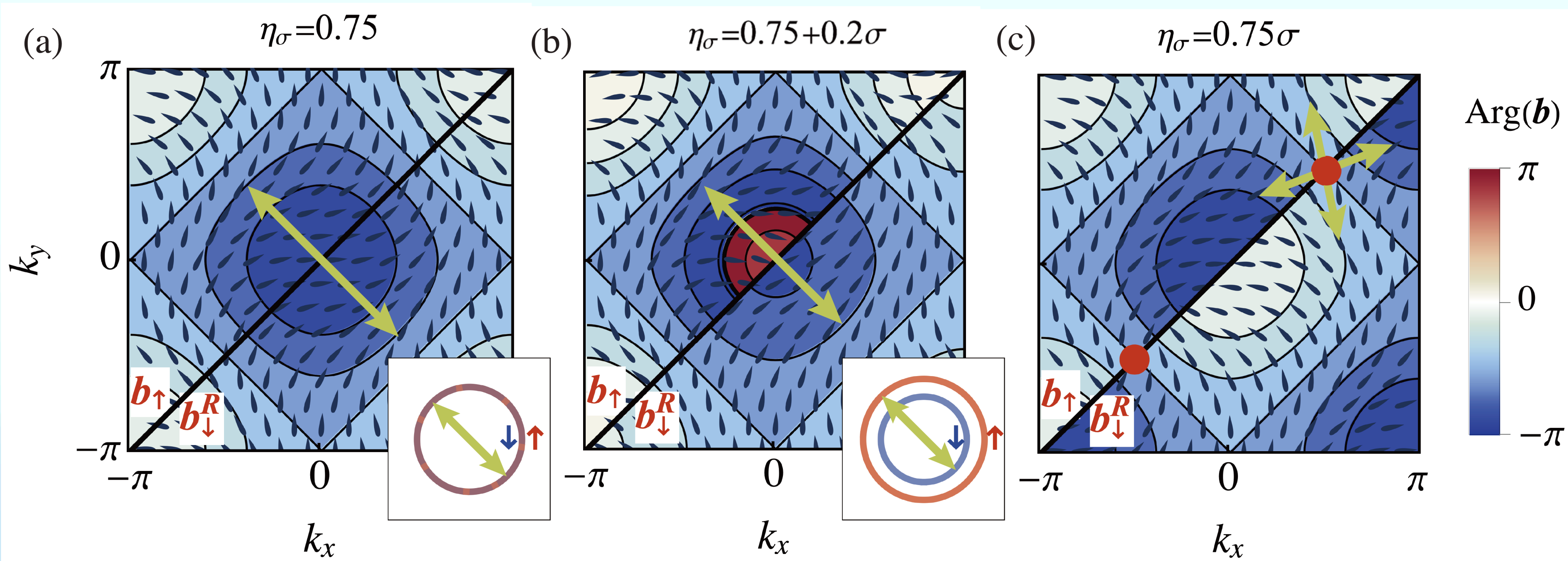
Similar nesting conditions

$$\boldsymbol{b}^R(\boldsymbol{k}+\boldsymbol{Q}) \parallel \boldsymbol{b}(-\boldsymbol{k})$$

$$\boldsymbol{b}^R\left(\frac{\boldsymbol{Q}}{2}+\frac{\boldsymbol{k}}{2}\right) \parallel \boldsymbol{b}\left(\frac{\boldsymbol{Q}}{2}-\frac{\boldsymbol{k}}{2}\right)$$



Superconducting QGN

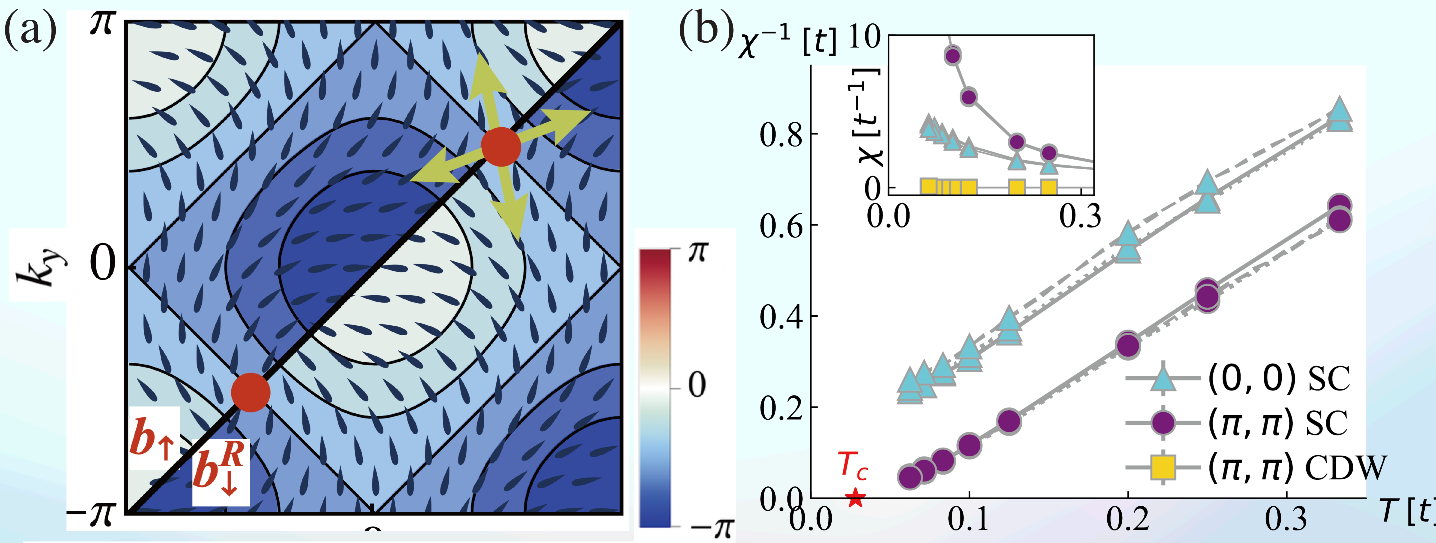


$\boldsymbol{Q} = \boldsymbol{0}$

$\boldsymbol{Q} \sim \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\boldsymbol{Q} = (\pi, \pi)$

QMC



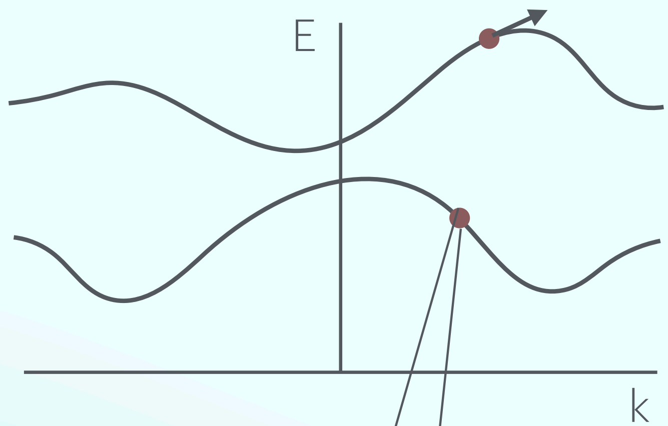
$\eta_\sigma = 0.75\sigma$

Summary

- We developed an algebraic approach to quantum geometric nesting
- It allows intuitive study of ordering instabilities of flat bands
- QMC validates the approach for a simple model system
- We also obtain relations between a “high temperature stiffness” and a generalized quantum metric (for both particle/hole and superconducting orders at any $\boldsymbol{Q}$)
- ?? Is there a way to include *non-symmetry breaking “orders”* like FQAHE?

Kinetic theory

Bands



$|\psi_{nk}\rangle$ → Geometry

$\epsilon_n(k)$ → $\mathbf{v}_n(k)$ $\sigma_{xx} \sim \langle v_n^x v_n^x \rangle$

Energetics

$\Omega_{n,\alpha\beta}(k)$ Berry curvature $\sigma_{xy} \sim \langle \Omega \rangle$

$g_{n,\alpha\beta}(k)$ Quantum metric

Souza, Wilkens, Martin 2000
Various suggestions

Role in interacting systems?

Linear response

A host of "conventional" (but still complex) transport coefficients

$$\mathbf{j} = \mathbf{L}^{11}\mathcal{E} + \mathbf{L}^{12}(-\nabla T),$$
$$\mathbf{j}^q = \mathbf{L}^{21}\mathcal{E} + \mathbf{L}^{22}(-\nabla T),$$

Ashcroft+Mermin

And some originating from Berry curvature

Anomalous Hall effect

$$\boldsymbol{j}_e^{ah} = e^2 \sum_n \int \frac{d^d \boldsymbol{k}}{(2\pi)^d} n_F(\omega_n(\boldsymbol{k}))\boldsymbol{\Omega}_n(\boldsymbol{k}) \times \boldsymbol{E}$$

Karplus+Luttinger, 1954!

Gyromagnetic effect

$$\alpha_{ij}^{\mathrm{GME}} = \frac{i\omega\tau e}{1 - i\omega\tau} \sum_n \int [d\boldsymbol{k}](\partial f/\partial \varepsilon_{\boldsymbol{k}n}) v_{\boldsymbol{k}n,i} m_{\boldsymbol{k}n,j}$$

Ma+Pesin 2015; Song *et al* 2016

Chiral anomaly...

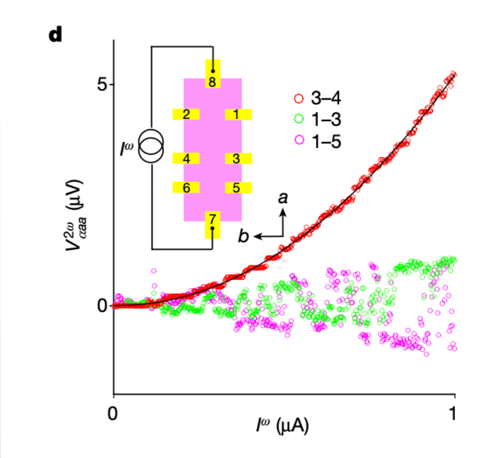
Non-linear response

Non-linear Hall effect

$$j_a^0 = \chi_{\text{abc}} \mathcal{E}_b \mathcal{E}_c^*, \; j_a^{2\omega} = \chi_{\text{abc}} \mathcal{E}_b \mathcal{E}_c,$$

$$\chi_{\text{abc}} = -\varepsilon_{adc} \frac{e^3 \tau}{2(1 + i\omega\tau)} \int_k f_0 (\partial_b \Omega_d)$$

Sodemann+Fu, 2015



Q. Ma *et al*, 2019 (WTe2)

Non-linear response

Quantum metric effects

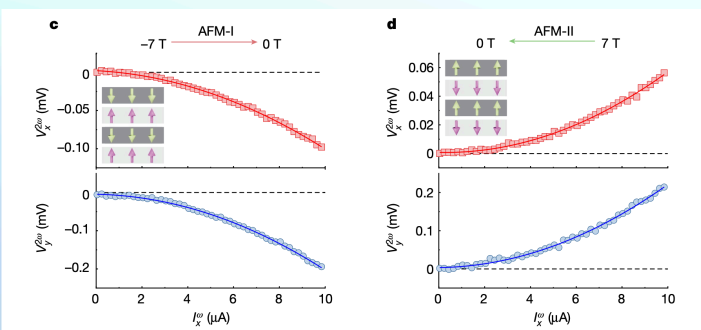
$$\sigma^{\mathrm{BCPD}}_{a;bc} = \frac{e^3}{\hbar} \sum_{m,p,\boldsymbol{k}} \int [d\boldsymbol{k}] f_m \left[\partial_a \tilde{\mathcal{G}}^{bc}_{mp} + \partial_b \tilde{\mathcal{G}}^{ac}_{mp} + \partial_c \tilde{\mathcal{G}}^{ab}_{mp}\right].$$

K. Das *et al*,2023

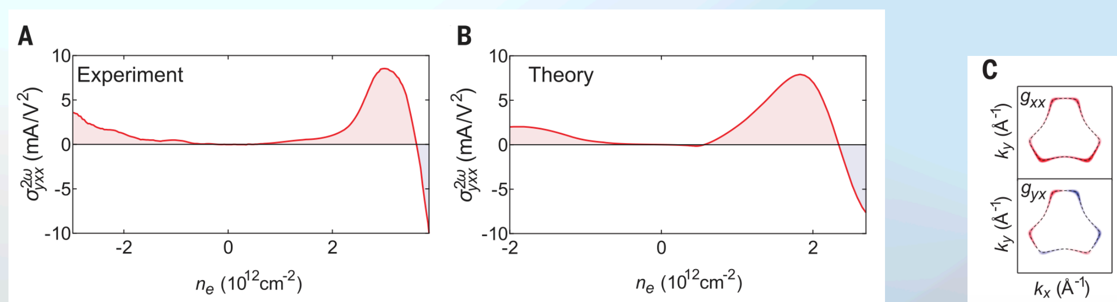
D. Kaplan *et al*, 2024

"Band-resolved quantum metric dipole"

$MnBi_2Te_4$



N. Wang *et al*, 2023



A. Gao *et al*, 2023

Response to field gradient

$$j^{\mu}_{\text{geom.}}(\mathbf{r}) = -\frac{Q^2}{2\hbar}\int \frac{d^D\mathbf{K}}{(2\pi)^D}\, f_0(\mathbf{K}) E^{(0)}_{\nu\lambda} \frac{\partial g^{\nu\lambda}(\mathbf{K})}{\partial K_\mu},$$

Quantum metric dipole

M. Lapa + T. Hughes, 2019

$$E_{\mu\nu} = \partial E_\mu / \partial x_\nu$$

Theoretical approaches

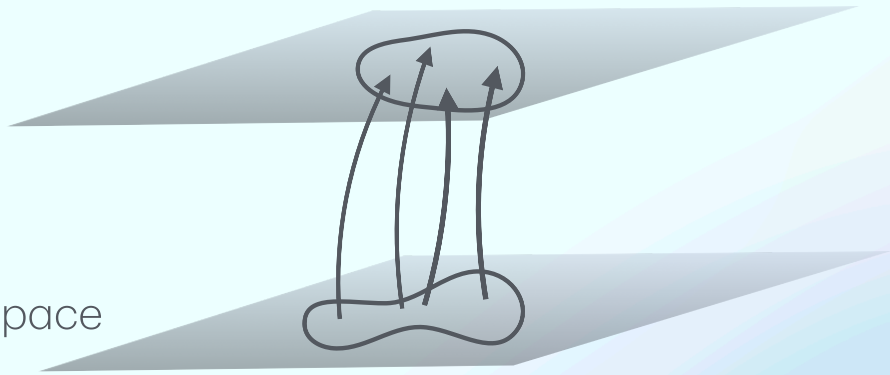
- Quantum response theory (~Kubo)
  - Usually regarded as most rigorous method
  - Formal. Simple results often obtained by very technical route.
  - Does not show dis-equilibrium
- Semi-classics
  - Intuitive
  - Directly reveals deviations from equilibrium
  - Approximate?

Semiclassics

- Boltzmann equation

$$\partial_t f_n + \frac{dx_\mu}{dt}\frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt}\frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

Incompressible flow of states in phase space



- One particle equations of motion

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}_n = \boldsymbol{\nabla}_{\boldsymbol{k}}\tilde{\epsilon}_{n\boldsymbol{k}} - \frac{d\boldsymbol{k}}{dt}\times\boldsymbol{\Omega}_{n\boldsymbol{k}},$$
$$\frac{d\boldsymbol{k}}{dt} = \boldsymbol{F}_n = -e\boldsymbol{E} - e\frac{d\boldsymbol{x}}{dt}\times\boldsymbol{B}.$$

Q. Niu ++

The power of semiclassics

- Anomalous Hall effect

$$\boldsymbol{j}_{ah} = \int_k f_n e \boldsymbol{v}_{ah} = -e \int_k f_n \frac{d\boldsymbol{k}}{dt} \times \boldsymbol{\Omega}_n = e^2 \boldsymbol{E} \times \int_k f_n \boldsymbol{\Omega}_n$$

Immediately gives AHE

- Thermal conductivity

$$f_n = n_F(\epsilon; T(x)) + g_n$$

$$g_n = \tau n'_F \boldsymbol{v}_n \cdot \frac{(\epsilon - \mu)\boldsymbol{\nabla} T}{T^2}$$

n.b. essential to include T(x).

$$j^q_\mu = \int_k (\epsilon_n - \mu) v^\mu_n g_n = -\frac{\partial_\nu T}{T^2} \tau \int_k n'_F (\epsilon - \mu)^2 v^\mu_n v^\nu_n$$

"Standard" thermal conductivity

The power of semiclassics

- Anomalous Hall effect

$$\boldsymbol{j}_{ah} = \int_k f_n e \boldsymbol{v}_{ah} = -e \int_k f_n \frac{d\boldsymbol{k}}{dt} \times \boldsymbol{\Omega}_n = e^2 \boldsymbol{E} \times \int_k f_n \boldsymbol{\Omega}_n$$

Immediately gives AHE

- Thermal conductivity

These and many other results agree with Kubo exactly. This is because the semiclassical approximation is exact to first order in spatial gradients.

$$f_n = n_F(\epsilon; T(x))$$

$$g_n = \tau n_F' \boldsymbol{v}_n \cdot \frac{(\epsilon - \mu)\boldsymbol{\nabla} T}{T^2}$$

n.b. essential to include T(x).

$$j_\mu^q = \int_k (\epsilon_n - \mu) v_n^\mu g_n = -\frac{\partial_\nu T}{T^2} \tau \int_k n_F' (\epsilon - \mu)^2 v_n^\mu v_n^\nu$$

"Standard" thermal conductivity

Semiclassics

- Boltzmann equation

$$\partial_t f_n + \frac{dx_\mu}{dt}\frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt}\frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

Incompressible flow of states in phase space

- One particle equations of motion

$$\begin{aligned}\frac{d\boldsymbol{x}}{dt} &= \boldsymbol{v}_n = \boldsymbol{\nabla}_{\boldsymbol{k}}\tilde{\epsilon}_{n\boldsymbol{k}} - \frac{d\boldsymbol{k}}{dt}\times\boldsymbol{\Omega}_{n\boldsymbol{k}},\\ \frac{d\boldsymbol{k}}{dt} &= \boldsymbol{F}_n = -e\boldsymbol{E} - e\frac{d\boldsymbol{x}}{dt}\times\boldsymbol{B}.\end{aligned}$$

Q. Niu ++

Just written down! $i\hbar\partial_t\rho = [H,\rho]$

Wavepacket approximation $i\hbar\partial_t\mathbf{x} = [H,\mathbf{x}]$

Semiclassics

- Boltzmann equation

$$\partial_t f_n + \frac{dx_\mu}{dt}\frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt}\frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

Incompressible flow of states in phase space

- One particle equations of motion

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}_n = \boldsymbol{\nabla}_{\boldsymbol{k}}\tilde{\epsilon}_{n\boldsymbol{k}} - \frac{d\boldsymbol{k}}{dt} \times \boldsymbol{\Omega}_{n\boldsymbol{k}},$$
$$\frac{d\boldsymbol{k}}{dt} = \boldsymbol{F}_n = -e\boldsymbol{E} - e\frac{d\boldsymbol{x}}{dt} \times \boldsymbol{B}.$$

Q. Niu ++

Just written down! $i\hbar\partial_t\rho = [H, \rho]$

Two semiclassical approximations at the same time

Wavepacket approximation $i\hbar\partial_t\mathbf{x} = [H, \mathbf{x}]$

Kinetics

Liouville-von Neumann equation $$i\hbar\partial_t\rho = \left[H, \rho\right]$$

Systematic derivation of Boltzmann-like equations

Intuitively $$f_n(k) \sim \langle n, k \,|\, \rho \,|\, n, k \rangle$$

Two issues:

- Track spatial dependence
- Off-diagonal terms in density matrix (inter-band coherence)

Wigner transformation

$$\mathsf{F}(k,X) = \int dx\, e^{ikx} \mathsf{F}(X + \frac{x}{2}, X - \frac{x}{2})$$

$$\left\langle X + \frac{x}{2} \middle| \rho \middle| X - \frac{x}{2} \right\rangle$$

Wigner-Weyl quantization

$$C = AB \quad \longleftrightarrow \quad C(k,X) = A(k,X) \star B(k,X)$$

Operators

Functions on phase space

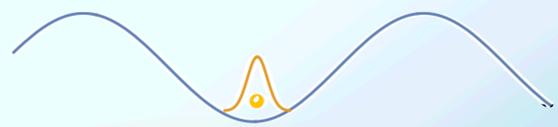
Moyal/star product

$$\star = \exp\left(\frac{i\hbar}{2}\left(\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_x\right)\right) = \exp\left(\frac{i\hbar}{2}\epsilon^{\alpha\beta}\overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta\right)$$

Symplectic form $[q_\alpha, q_\beta] = i\hbar\epsilon^{\alpha\beta}$

Semi-classical expansion

$$\star = \exp\left(\frac{i\hbar}{2}\epsilon^{\alpha\beta}\overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta\right) = 1 + \frac{i\hbar}{2}\epsilon^{\alpha\beta}\overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta + \cdots$$



Slow spatial variation

Systematic procedure to carry out gradient expansion while working in both position and momentum.

Star diagonalization

Even with phase space formalism, **H, F**, etc are still matrices

To pass to a semi-classical kinetic equation, we want to eliminate off-diagonal terms.

Diagonalization is compatible with star product

$$\mathsf{H} = U \star \tilde{h} \star U^\dagger \qquad \mathsf{F} = U \star \tilde{f} \star U^\dagger$$

Formal diagonalization carried out order by order in gradient expansion.

Almost the desired diagonal Hamiltonian and distribution

BUT we need to ensure gauge invariance.

Gauge invariance

Quantum mechanics allows an arbitrary phase for each eigenfunction. Within a band this is the usual origin of Berry gauge field.

$$U \to U \star E^{i\theta(x,k)}$$

$$\tilde{f} \to E^{-i\theta} \star \tilde{f} \star E^{i\theta}$$

Transforms non-trivially because of star product

Gauge invariant form

$$f_n = \mathrm{tr}\left[U \star p_n \tilde{f} \star U^\dagger\right]$$

Projector on n^{th} band

Energy density

$$\rho_n^E = \frac{1}{2}\,\mathrm{tr}[U \star p_n(\tilde{h} \star \tilde{f} + \tilde{f} \star \tilde{h}) \star U^\dagger]$$

Gauge invariance

We worked these out to second order in gradients:

$$f_n = \tilde{f}_n + \hbar\varepsilon_{\alpha\beta}\partial_\alpha(\tilde{f}_n A_{n\beta}) + \frac{\hbar^2}{2}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial^2_{\alpha\sigma}(\tilde{f}_n(\Lambda_\beta\Lambda_\lambda)_{nn})$$

$$h_n \equiv \tilde{h}_n + \hbar\varepsilon_{\alpha\beta}\partial_\alpha\tilde{h}_n\left(A_{n\beta} + \hbar\varepsilon_{\sigma\lambda}A_{n\lambda}\partial_\sigma A_{n\beta}\right) - \hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial^2_{\alpha\sigma}\tilde{h}_n\left(\frac{1}{4}\left\{\Lambda_\beta,\Lambda_\lambda\right\}_{nn} - A_{n\beta}A_{n\lambda}\right)$$

$$\Lambda_\alpha = -iU^\dagger \star \partial_\alpha U$$

$$A_\alpha = \mathrm{diag}\left(\Lambda_\alpha\right)$$ Berry gauge field

Now simply

$$\mathcal{N} = \int_{x,p}\sum_n f_n$$ f_n is electron density in a band

$$E = \int_{x,p}\sum_n f_n h_n$$ $f_n h_n$ is energy density in a band

Kinetic equation

$$\partial_t f_n + \partial_\alpha \mathcal{J}_\alpha = 0$$

Just continuity equation for density in a band

c.f. simple convection $\mathcal{J}_\alpha = f_n v_\alpha$ → (Incompressible flow) Boltzmann equation

Here, to second order:

$$\begin{aligned}\mathcal{J}_\alpha =& \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu} h \partial_\beta g_{\nu\lambda}\right)\right] \\ &+ \hbar^2 \varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta} h g_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu} h\right]\end{aligned}$$

Kinetic equation

$$\partial_t f_n + \partial_\alpha \mathcal{J}_\alpha = 0$$

Just continuity equation for density in a band

c.f. simple convection $\mathcal{J}_\alpha = f_n v_\alpha$ → Boltzmann equation

Incompressible flow

Here, to second order:

$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu} h \partial_\beta g_{\nu\lambda}\right)\right]$$

$$+ \hbar^2 \varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta} h g_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu} h\right]$$

quantum metric $g_{\alpha\beta} = \frac{1}{2}\mathrm{diag}\left(\Lambda_\alpha\Lambda_\beta + \Lambda_\beta\Lambda_\alpha\right) - A_\alpha A_\beta = \frac{1}{2}\,\mathrm{tr}(\partial_\alpha P_n \partial_\beta P_n)$

Kinetic equation

$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h\Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}h\partial_\beta g_{\nu\lambda}\right)\right]$$
$$+ \hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h\partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta}hg_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu}h\right]$$

- General and exact up to second order in semiclassical expansion
- Explicitly contains real space and momentum space, and capable of describing inhomogeneous systems. Magnetic field can be included by modifying $\epsilon_{\alpha\beta}$
- Quantum geometry of bands appears explicitly, takes local band Hamiltonian as input
- Only band-intrinsic quantities appear: but these are bands renormalized by quantum corrections.
- Ready to attack all sorts of problems!

Example: separable problem

$$\mathsf{H}(x,p) = \mathsf{H}_0(p) - eV(x)\mathsf{I}_N$$

Band form Diagonal potential

Kinetic equation (relaxation time approx)

$$\begin{aligned}\partial_t f =& -\partial_{x_i}\left[f\left(v_i + \frac{e}{\hbar}\Omega_{ij}\partial_{x_j}V + e\mathsf{T}_{ij}\partial^2_{tx_j}V + \frac{e^2}{\hbar}\left(\partial_{k_j}\mathsf{T}_{il} - \frac{1}{2}\partial_{k_i}\mathsf{T}_{jl}\right)\partial_{x_j}V\partial_{x_l}V + \frac{e}{\hbar}\left(\hbar v_j\mathsf{T}_{il} - \frac{1}{2}\partial_{k_i}\mathsf{g}_{jl}\right)\partial^2_{x_jx_l}V\right)\right] \\ & - \frac{e}{\hbar}\partial_{k_i}f\partial_{x_i}V + \frac{e}{2\hbar}\partial^2_{x_ix_l}(f\partial_{k_j}\mathsf{g}_{il}\partial_{x_j}V) - \frac{e}{\hbar}\partial^2_{k_ix_j}(f\mathsf{g}_{jl}\partial^2_{x_ix_l}V) + \frac{1}{24\hbar}\partial^3_{x_ix_jx_l}f\partial^3_{k_ik_jk_l}\varepsilon \\ & + \frac{e}{24\hbar}\partial^3_{k_ik_jk_l}f\partial^3_{x_ix_jx_l}V + \mathcal{I}[f] + \mathcal{O}(\partial_x^4).\end{aligned} \tag{3.29}$$

$$T_{n;\mu\nu} = -\sum_{m\neq n}\frac{\langle\partial_{p_\mu}u_n|u_m\rangle\langle u_m|\partial_{p_\mu}u_n\rangle + (\mu\leftrightarrow\nu)}{\epsilon_n - \epsilon_m}$$

"band-normalized quantum metric"

(*not* a purely geometric quantity)

Non-equilibrium current

Linear conductivity: q dependence $\sigma_{ij}^{(1)}(q,\omega) = \sigma_{ij}^{(1,0)}(\omega) + \sigma_{ijl}^{(1,1)}(\omega) i q_l + \mathcal{O}(q^2)$

$$\sigma_{ijl}^{(1,1)}(\omega) = \sum_{n=1}^{N} e^2 \int_k \left(f_n' \frac{v_i v_j v_l}{(i\omega - \tau^{-1})^2} + f_n \frac{1}{2}(v_j \mathsf{T}_{li} + v_l \mathsf{T}_{ij} - \underline{\hbar^{-1}\partial_{k_j}\mathsf{g}_{li} - \hbar^{-1}\partial_{k_l}\mathsf{g}_{ij}}) - f_n \frac{\hbar^2}{2}(M_{j;il} + M_{l;ij}) \right)$$

Lapa+Hughes

Quadratic conductivity

$$\sigma_{ijl}^{(2)}(q\,\omega; q'\omega') = \frac{e^3}{\hbar^2} \sum_{n=1}^{N} \int_k f_n \left[\frac{\partial^3_{k_i k_j k_l}\varepsilon}{(i\omega - \tau^{-1})(i\omega' - \tau^{-1})} - \underline{\left(\frac{\hbar v_i}{2} \mathsf{T}_{jl} - \partial_{k_i}\mathsf{T}_{jl} + \partial_{k_j}\mathsf{T}_{il} \right)} + \underline{\frac{\partial_{k_j}\Omega_{ij}}{i\omega' - \tau^{-1}}} \right] + \mathcal{O}(q)$$

Band-normalized quantum metric dipole

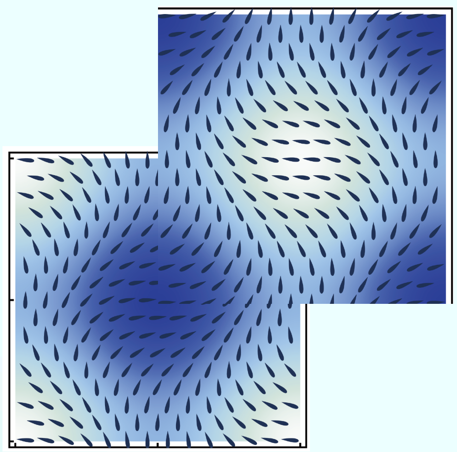
Sodemann+Fu

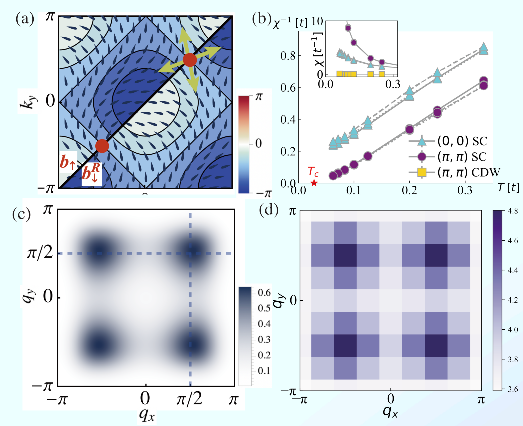
Summary

$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h\Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}h\partial_\beta g_{\nu\lambda}\right)\right]$$
$$+ \hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h\partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta}hg_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu}h\right]$$

- We found a procedure to obtain semiclassical electron dynamics organized by spatial gradients
- We found a general result for the phase space current to *second* order in gradients, capable of handling inhomogeneous systems and magnetic fields
- Prior results for linear and non-linear response are recovered, and in some cases corrected
- Still needed: a scattering/interactions theory with the same validity

Thanks





$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu} h \partial_\beta g_{\nu\lambda}\right)\right]$$
$$+ \hbar^2 \varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta} h g_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu} h\right]$$