An analog of topological entanglement entropy for mixed states[1]

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Main results

We propose the **convex-roof extension** of quantum conditional mutual information ("co(QCMI)") as a diagnostic of long-range entanglement in a mixed state. We focus primarily on topological states subjected to **local decoherence**, and employ the Levin-Wen scheme to define co(QCMI), so that for a pure state, co(QCMI) equals topological entanglement entropy (TEE). By construction, co(QCMI) is zero if and only if a mixed state can be decomposed as a convex sum of pure states with zero TEE.

We show that co(QCMI) is non-increasing with increasing decoherence when Kraus operators are proportional to the product of onsite unitaries. For the 2d toric code decohered by onsite bit/phase-flip noise, we show that co(QCMI) is non-zero below the error-recovery threshold and zero above it. We conjecture and provide evidence that in this example, co(QCMI) equals TEE of a recently introduced pure state [2]. We develop a tensor-assisted Monte Carlo (TMC) computation method to efficiently evaluate the Rényi TEE for the aforementioned pure state and provide non-trivial consistency checks for our conjecture. We use TMC to also calculate the universal scaling dimension of the anyon-condensation order parameter at this transition.

Model

We investigate the toric code model ground state ρ_0 subjected to local decoherence in the form of phase-flipped error with error rate p. More specifically, we focus on one decomposition for the decohered mixed state $\rho(p)$ proposed in Ref. [2] given as

$$\rho(p) = \sum_{g_x} |\psi_{g_x}(\beta)\rangle\langle\psi_{g_x}(\beta)|,$$

where $|\psi_{g_x}(\beta)\rangle = g_x|\psi(\beta)\rangle$, g_x is product of Pauli-X operators that form closed loops, and the inverse temperature β is related to error rate p as $\tanh(\beta) = 1-2p$. The pure state $|\psi(\beta)\rangle$ is given by

$$|\psi(\beta)\rangle \propto \sum_{x_e} \sqrt{Z_{x_e}(\beta)}\,|x_e\rangle,$$

where $Z_{x_e}(\beta) = \sum_{z_v} e^{\beta \sum_e x_e \prod_{v\in e} z_v}$ is the partition function of the 2d Ising model with bond strengths given by x_e.

Since states $|\psi_{g_x}(\beta)\rangle$ are related to the state $|\psi(\beta)\rangle$ via onsite unitaries. Such decomposition implies that $\mathrm{co(QCMI)}[\rho(p)] \le \mathrm{TEE}(|\psi(\beta)\rangle)$.

Numerical Result

$\gamma \approx \ln 2$ at low temperature for all system sizes, is monotonically non-increasing as p increases, and tends towards zero as $T \to T_c$. Further, as the system size is increased, γ tends towards $\ln 2$ at a relatively higher temperature and is also non-zero up till a relatively higher temperature (i.e., the range of decoherence rate over which the topological phase is visible in a finite system increases).

Perhaps more interestingly, they strongly suggest that as one approaches the critical point, so that $L \ll \xi$, Rényi co(QCMI) approaches zero. This is in strong contrast to (pure) ground state phase transition in toric code that is driven by a magnetic field, where in the critical regime, QCMI exceeds the TEE of the topological phase.

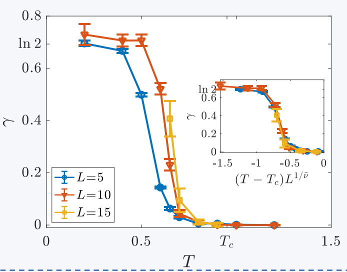

FIG. 2. **Result for Rényi TEE γ using Levin-Wen scheme.** Rényi TEE γ for the state $|\psi(\beta)\rangle$ against temperature $T = (\beta^{-1})$ and the rescaled temperature $(T-T_c)L^{1/\tilde{\nu}}$ (inset) with $T_c = 0.951$ and $\tilde{\nu} \approx 3.2$.

co(QCMI)

A mixed state ρ can be expressed as convex sum of pure states, that is, $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, where $p_i \ge 0$, and $\sum_i p_i = 1$.

For any function f defined on pure states, the convex-roof extension of f on mixed states is defined as:

$$\mathrm{co}(f)[\rho] = \inf\left(\sum_i p_i\, f(|\psi_i\rangle) \mid \rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, p_i \ge 0, \sum_i p_i = 1\right)$$

In our study, we focus on such extension of quantum conditional mutual information under the Levin-Wen scheme, which measures the **topological entanglement entropy** (TEE) in pure states. We show that,

1. The co(QCMI) for local decoherence where Kraus operators are proportional to a unitary is monotonically non-increasing as a function of the decoherence rate.
2. For toric code under local phase-flip or bit-flip decoherence in any dimension, the co(QCMI) must be non-zero in the regime where error-correction is feasible. Under certain assumptions, we also argue that the value of co(QCMI) equals the TEE of the pure toric code.

Tensor-assisted Monte Carlo (TMC)

We computed TEE via co(QCMI) of Rényi entanglement entropy, namely, $\gamma = \frac{1}{2}S_2(A{:}B|C) = \frac{1}{2}\big(S_2(AC)+S_2(BC)-S_2(C)-S_2(ABC)\big)$, with subregions A, B and C defined in Fig. 1 (b).

The second order Rényi entropy on subregion A and its complement $B=\bar{A}$ is defined as $S_2(A) = -\ln \mathrm{tr}\,\rho_A^2$. For the pure state $|\psi(\beta)\rangle$, one finds,

$$\mathrm{tr}\,\rho_A^2 = \frac{\sum_{x_e,x_e'} \sqrt{Z_{x_A',x_B} Z_{x_A,x_B'} Z_{x_A,x_B} Z_{x_A',x_B'}}}{\sum_{x_e,x_e'} Z_{x_e} Z_{x_e'}} = \left[\sqrt{\frac{Z_{x_A',x_B} Z_{x_A,x_B'}}{Z_{x_A,x_B} Z_{x_A',x_B'}}}\right]$$

where $[\,\cdot\,]$ denotes the weighted average over bond configurations with the joint probability proportional to the corresponding partition functions Z_{x_e} and $Z_{x_e'}$. $Z_{x_A',x_B}, Z_{x_A,x_B'}$ are simply Ising partition functions with bonds in region A swapped between the two replicas.

The calculation of the mixed state co(QCMI) is mapped to the random bond Ising model along the **Nishimori line** [3]. That is, with the Nishimori condition $\tanh(\beta) = 1-2p$, sampling bond configuration according to the partition function Z_{x_e}, is equivalent to having anti-ferromagnetic bond according to the binomial distribution with probability p, followed by gauge transformation $x_e \to x_e \prod_{v\in e}\sigma_v$ with $\sigma_v = \pm 1$ on every site.

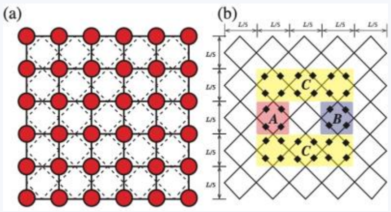

FIG. 1. **Tensor formalism for the Ising model partition function.** (a) A tensor network for 5×5 system, with $(L+1)^2 = 36$ local tensors. Each tensor encodes the interaction between 4 Ising spins, with each leg containing the local spin degree of freedom ($d=2$). The dashed lines represent the original lattice. (b) The Levin-Wen scheme in the tilted square lattice. Here L is a multiple of 5, and we divide the system into a 5×5 grid, and choose the subregions as depicted, similar to Ref. [4].

However, for small error rate p, the direct sampling of the above form is exponentially hard, because the square root term is around 1 only for exponentially small portion of the total configurations, and 0 otherwise.

We tackle this problem with the TMC algorithm, by combining the non-equilibrium Monte Carlo sampling to convert the exponential observables to a uniform work done, with tensor contraction to compute each partition function.

Consider

$$Q(\lambda) = \sum_{x_e,x_e'} Z_{x_e} Z_{x_e'} \left(\frac{Z_{x_A',x_B} Z_{x_A,x_B'}}{Z_{x_A,x_B} Z_{x_A',x_B'}}\right)^{\lambda/2} = \sum_{x_e,x_e'} Z_{x_e} Z_{x_e'}\, g(x_e,x_e',\lambda)$$

as the partition function connecting the numerator and denominator in the above equation of trace. From Jarzynski equality [5], the exponential of free energy difference equals the weighted average of the exponential of the work done over all realizations bringing the system from $Q(0)$ to $Q(1)$:

$$e^{-S_2} = \frac{Q(1)}{Q(0)} = e^{\Delta F} = \langle e^{W}\rangle, \text{ with}$$

$$W = \int_0^1 d\lambda\, \frac{\partial \ln g(x_e,x_e',\lambda)}{\partial\lambda} = \int_0^1 d\lambda \ln\sqrt{\frac{Z_{x_A',x_B} Z_{x_A,x_B'}}{Z_{x_A,x_B} Z_{x_A',x_B'}}}$$

The observable now is log of the original square root term which has a much better distribution and can be sample well in polynomial time.

At each Monte Carlo step, one samples bond configurations in two replicas according to $Q(\lambda)$, gradually tunes λ from 0 to 1, and records the work done dW. To find the acceptance ratio and perform the measurement, Ising partition functions need to be evaluated, and they are carried out by contracting corresponding tensor shown in Fig. 1. (a).

Bibliography

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