# Entanglement entropy in quantum phase transitions

Menghan Song,[1] Jiarui Zhao,[1] Meng Cheng,[2] Cenke Xu,[3] Michael M. Scherer,[4] Lukas Janssen,[5] and Zi Yang Meng[1], *
1 Department of Physics, The University of Hong Kong
2 Department of Physics, Yale University
3 Department of Physics, University of California, Santa Barbara
4 Institute for Theoretical Physics III, Ruhr-University Bochum
5 Institut für Theoretische Physik and Würzburg-Dresden, TU Dresden

## Main results

We explore the finite size scaling behavior of Rényi entanglement entropy (EE) in various two-dimensional quantum many-body systems, focusing on the nature of quantum phase transitions, particularly the deconfined quantum critical point (DQCP) ①②.

Our findings reveal that **(1)** For the (2 + 1)d O(3) Wilson-Fisher QCP, we observe logarithmic corrections at sharp corners but not for smooth boundaries. **(2)** In the context of DQCP, specifically the square-lattice SU(N) models, we demonstrate that anomalous logarithmic correction from smooth boundary cut persists for N below a critical threshold $N_c$, indicating that these transitions do not belong to conformal fixed points. For N above this threshold (between 7 and 8), the DQCPs align with conformal fixed points, describable by Abelian Higgs field theories. **(3)** For the corner cut, we observe no logarithmic correction for $N < N_c$ but universal logarithmic corrections consistent with the free gaussian value for $N > N_c$ .

## (2+1)d O(3) QCP

The (2+1)d O(3) phase transition can be realized in a bilayer Heisenberg model defined on a bilayer square lattice with nearest-neighbor antiferromagnetic intra-layer coupling $J$ and inter-layer coupling $J_\perp$, as shown in FIG. 2 (a). The slope of subtracted EE $S_A^{(2)}(2l) - 2S_A^{(2)}(l)$ versus $\ln l$ represents the magnitude of $s_c(\theta)$. As shown in (d), for the smooth boundary at QCPs, we get $s_c(\pi) = 0$ , for four $\pi/2$ corners we get previously determined value ~0.08.

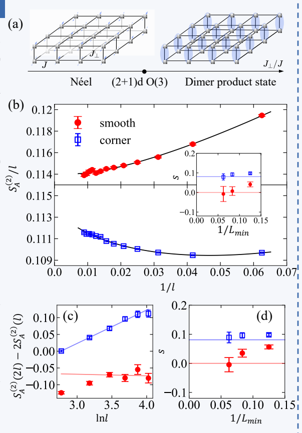


FIG. 2: (a) lattice of bilayer Heisenberg model. (b) EE for both smooth boundary and boundary with corners for different boundary lengths. The black line is fitted from Eq. ❶ directly. (c) The subtracted EE versus $\ln l$. (d) The fitted $s$ from data in (c) with respect to the smallest retained system size $1/L_{min}$ in the fitting process.

## Scaling behavior of EE

The scaling of EE ($S = al - s\ln l - b$ ❶) follows the "area law" with the leading order be the "area" of the entanglement boundary. The logarithmic sub-leading term encodes universal information at quantum phase transition points (QCPs). The detailed scaling behavior is summarized in TAB.1. In our work, we focus on the d=2 QCP case and study the sub-leading log-correction $s_c(\theta)$ comes from the geometric sharp corners of entanglement region. The positivity criteria of $s_c(\theta)$ can help to distinguish non-CFT QCPs.

| | | |
|---|---|---|
| d=1 CFT | $S \sim c\ln(l)$ | Heisenberg chain |
| **d=2 QCP** | $S \sim al - s_c\ln(l) - b$ | **O(N) Wilson-Fisher** |
| SSB | $S \sim al + s_G\ln(l)$ | Neel phase, Superfluid |
| Topological order | $S \sim al - \gamma_{top}$ | $Z_2$ top. order, Kitaev QSL |
| Fermi surface | $S \sim l\ln(l) + al - \cdots$ | Free fermion |

TAB.1 The scaling behavior of EE at different scenarios.

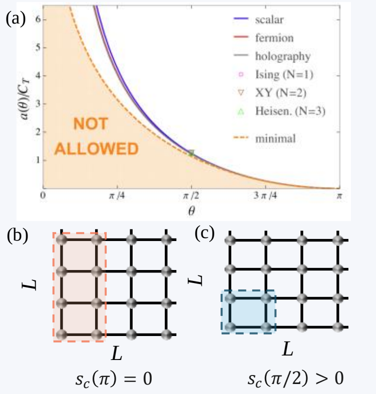


FIG.1 (a) Lower bound for $s_c(\theta)$ for a CFT, adapted from FIG.2 in **P. Bueno, PRB 93, 045131**. (b) and (c) shows the lattice bipartition for $\theta = \pi$ and $\pi/2$ respectively.

## SU(2) DQCP with smooth cut

If the DQCP is indeed a unitary CFT, the coefficient $s$ of the logarithmic correction must follow the constraint $s_c(\pi) = 0$. We measure the 2nd Rényi EE in both the J-$Q_2$ and J-$Q_3$ model with the following Hamiltonians and the lattice is shown in FIG.3 (a),

$$H_{J-Q_2} = -J\sum_{\langle ij\rangle} P_{ij} - Q\sum_{\langle ijkl\rangle} P_{i,j}P_{k,l}$$
$$H_{J-Q_3} = -J\sum_{\langle ij\rangle} P_{ij} - Q\sum_{\langle ijklmn\rangle} P_{i,j}P_{k,l}P_{m,n}$$

where $P_{ij}$ is the two-spin singlet projector. A DQCP is reached via increasing Q from the Néel ordered state. At DQCP, subtracted EE exhibits linear scaling against $\ln l$. The fitted slope of FIG.3 (c) with all available data points is found to be $s = -0.224(5)$ for the J-$Q_3$ model and $s = -0.289(6)$ for the J-$Q_2$ model. We have also examined the effect of changing $L_{min}$. As shown in the inset of FIG. 3(b) and FIG. 3 (d), $s$ for the DQCP in the J-$Q_3$ model seems stable against $L_{min}$, while the one for the J-Q2 model drifts slightly as $L_{min}$ increases. Our results show that even for subregions without sharp corners, within the available system size, the scaling of Rényi EE at the Néel-to-VBS DQCP still **has a logarithmic correction** to the leading perimeter law scaling.

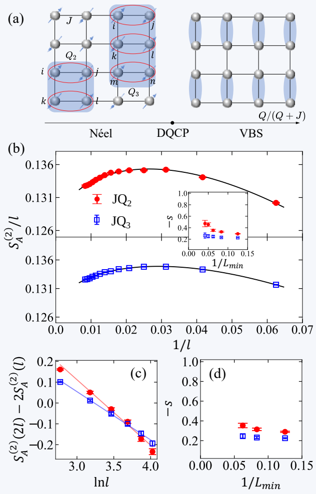


FIG. 3: (a) lattice of JQ model. (b) EE for smooth boundary cut. The black line is fitted from Eq. ❶ directly. (c) The subtracted EE versus $\ln l$. (d) The fitted $s$ from data in (c) with respect to the smallest retained system size $1/L_{min}$ in the fitting process.

## SU(N) DQCP

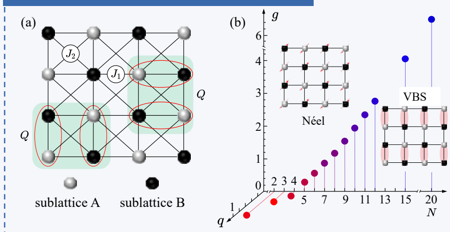


FIG. 4: (a) lattice of SU(N) $J_1J_2Q$ model. (b) The phase diagram of SU(N) $J_1J_2Q$ model. DQCP is reached by either tunning $q = Q/(Q + J_1)$ with $J_2 = 0$ at $N < 5$ or $g = J_1/J_2$ with $Q = 0$ at $N \geq 5$.

We study the SU(N) spin model defined in a Hilbert space of $N$ local states (colors) at each site of the square lattice as show in FIG. 4 (a) with $H = -\frac{J_1}{N}\sum_{\langle ij\rangle} P_{ij} - \frac{J_2}{N}\sum_{\langle\langle i,j\rangle\rangle} \Pi_{ij} - \frac{Q}{N^2}\sum_{\langle ijkl\rangle} P_{ij}P_{kl}$. Our data find a critical $N_c \approx 8$ below which $s_c(\pi) \neq 0$ and $s_c(\pi/2) = 0$ at DQCPs, which **violates the CFT constraint**. Above $N_c$, we find $s_c(\pi) = 0$ and $s_c(\pi/2) \propto$ Gaussion value $+ const$. Therefore, we conclude that DQCP below $N_c$ cannot be described by a CFT and thus are not continuous phase transition. Transitions above $N_c$ are possible candidates for the **genuine DQCPs**.

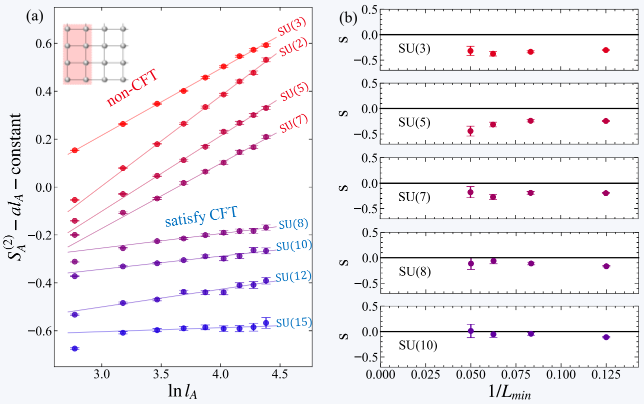


FIG. 5: Smooth cut: (a) EE minus the leading contribution such that the slope reflects the sub-leading coefficient $s$. (b) The fitted $s$ from data in (a) with respect to the smallest retained system size $1/L_{min}$ in the fitting process.

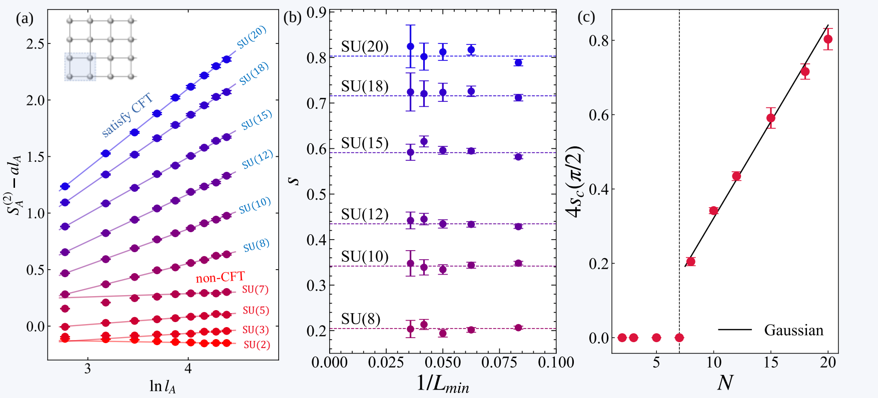


FIG. 6: Corner cut: (a) EE minus the leading contribution such that the slope reflects the sub-leading coefficient $s$. (b) The fitted $s$ from data in (a) with respect to the smallest retained system size $1/L_{min}$ in the fitting process. (c) The change of corner log-coefficient as a function of $N$. The black line is $s = 0.00648 \times 4 \times 2N + const$, where 0.00648 is the value for a free scalar field at a single 90° corner.

## Bibliography

① SciPost Phys. 17, 010 (2024), Menghan Song, Jiarui Zhao, Zi Yang Meng, Cenke Xu, Meng Cheng
② arXiv:2307.02547, Menghan Song, Jiarui Zhao, Meng Cheng, Cenke Xu, Michael M. Scherer, Lukas Janssen, Zi Yang Meng