Content

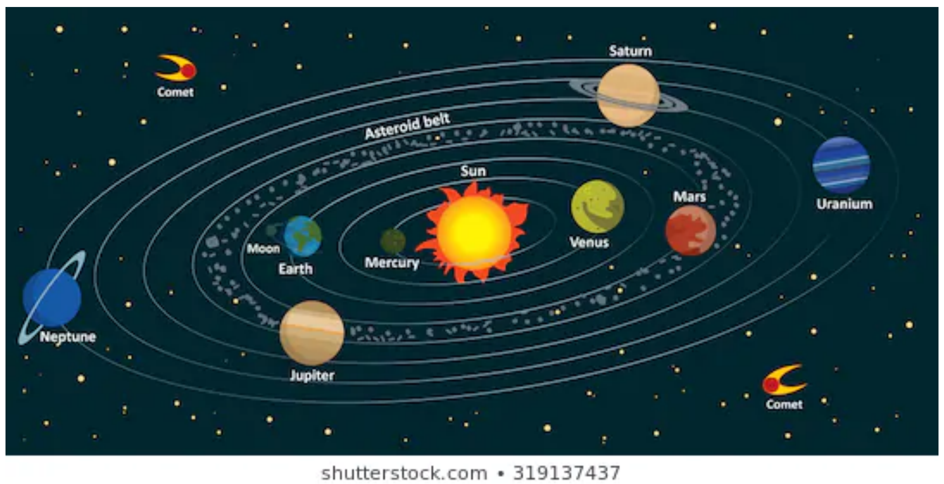
0. Introduction

1. Differential equations
  1.1 Classical equation of motion (classical mechanics, pendulum)
  1.2 Partial differential equation relaxation methods (electromagnetism, diffusion)
  1.3 Partial differential equation in space-time (traffic flow, tsunami)

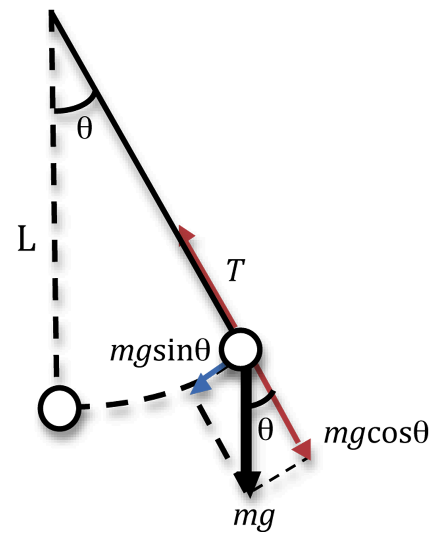
2. Eigenvalue problem
  2.1 Schrödinger equation and Hamiltonian (Harmonic oscillator, wave package)
  2.2 Quantum lattice model and Hibert space (Heisenberg model)
  2.3 Exact diagonalization of spin chain (Spin wave, Haldane conjecture, topology)
  2.4 Matrix product state and density matrix renormalization group (DMRG)

Differential equations

- Initial value problems: time-dependent equations with given initial conditions



Solar system



Pendulum

浮世绘，葛饰北斋，神奈川冲浪里

- Boundary value problems: differential equations with specific boundary values

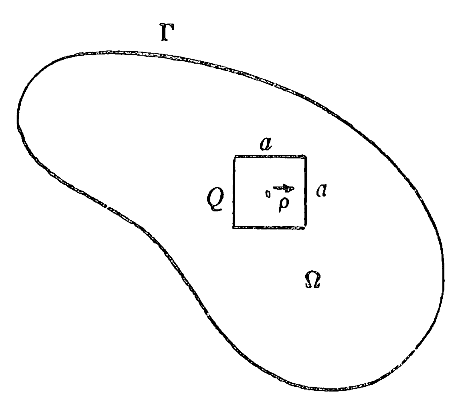
- Eigenvalue problems

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

- Am. Math. Mon. 73, 1 (1966)



Eigenvalues of Dirichlet problem for Laplacian

$$\tfrac{1}{2}\nabla^2 U + \lambda U = 0 \text{ in } \Omega,$$
$$U = 0 \text{ on } \Gamma.$$

Length of circumference

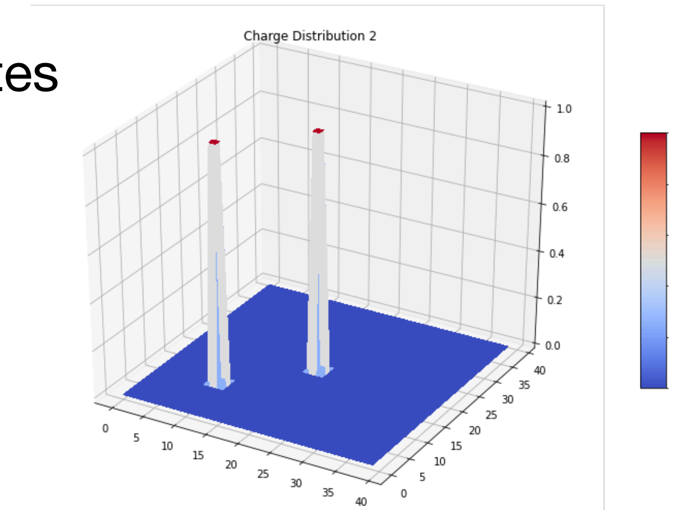
$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + \frac{1}{6}$$

Number of holes

$$(1-r)\frac{1}{6}$$

Partial differential equations

1st, 2nd derivatives of spatial and time coordinates



- Poisson equation

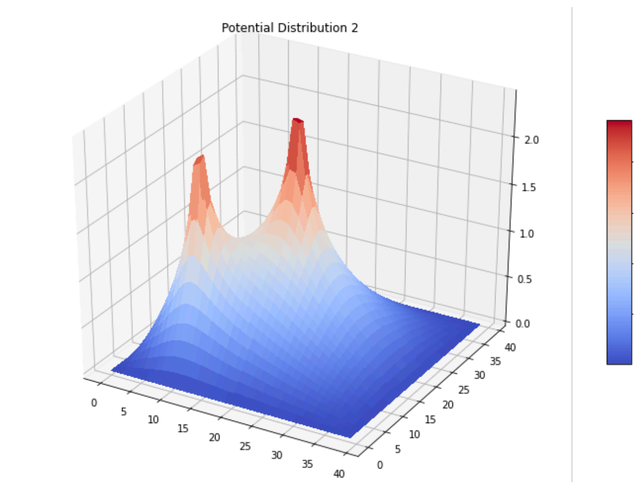
$$\Delta\phi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})$$

Laplacian $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$

$\rho(\vec{r})$ charge density inside domain V

$\phi(\vec{r})$ electrostatic potential

elliptic PDE



Dirichlet boundary condition $\phi(\vec{r}), \vec{r} \in \partial V$

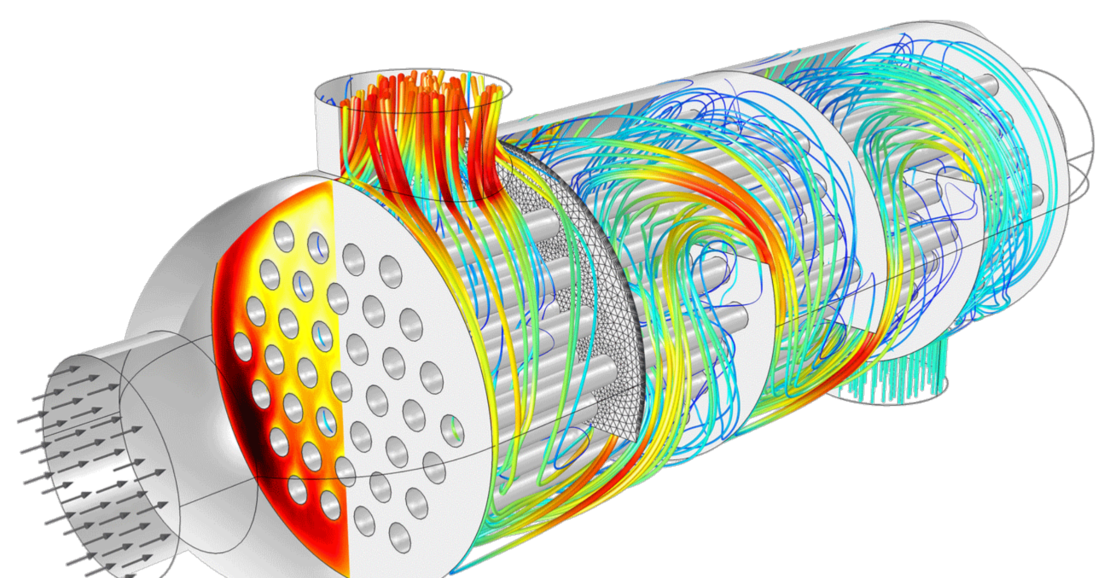
Neumann boundary condition $(\vec{n} \cdot \vec{\nabla})\phi(\vec{r}), \vec{r} \in \partial V$

- Diffusion equation

$$\frac{\partial u(\vec{r}, t)}{\partial t} - D\Delta u(\vec{r}, t) = S(\vec{r}, t)$$

$u(\vec{r}, t)$ concentration of a substance at position $\vec{r}$ and time t

$S(\vec{r}, t)$ source/drain D diffusion coefficient



parabolic PDE asymmetrical under time-reversal $t \to -t$

Cauchy initial value problem $u(\vec{r}, t = 0)$ on domain V

Neumann boundary condition $(\vec{n} \cdot \vec{\nabla})u(\vec{r}) = 0, \vec{r} \in \partial V$

Initial configuration must be consistent with the boundary condition

- Wave equation $$\frac{1}{c^2}\frac{\partial^2 u(\vec{r},t)}{\partial t^2} - \Delta u(\vec{r},t) = S(\vec{r},t)$$

c wave velocity

Hyperbolic PDE symmetrical under time-reversal, oscillations $t \to -t$

initial value problem $u(\vec{r}, t=0), \frac{\partial u(\vec{r},t)}{\partial t}|_{t=0}$

Initial configuration must be consistent with the boundary condition

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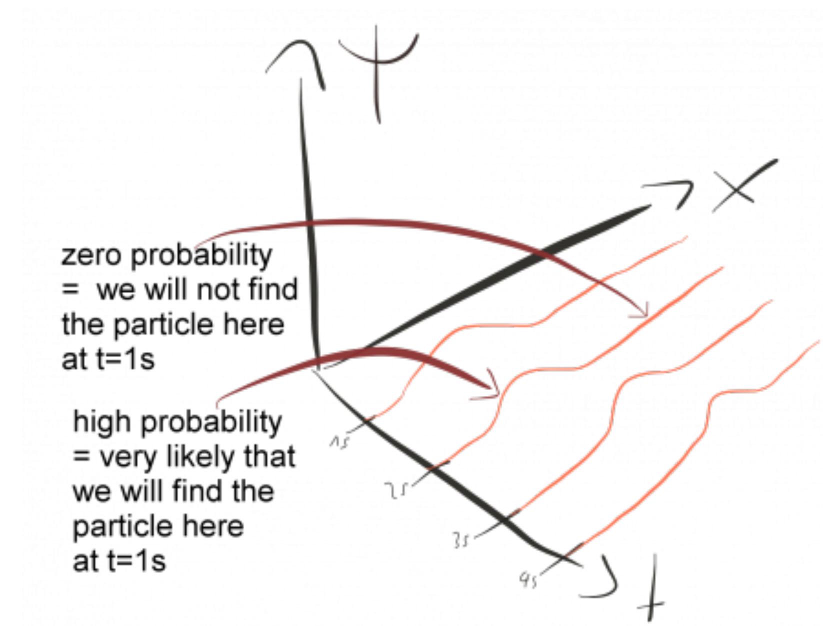
- Schrödinger equation $$i\hbar\frac{\partial \Psi(\vec{r},t)}{\partial t} = H\Psi(\vec{r},t)$$

For free particle $H = -\frac{\hbar^2}{2m}\Delta$

diffusion equation in imaginary time

- Fluid Dynamics, Navier-Stokes equation

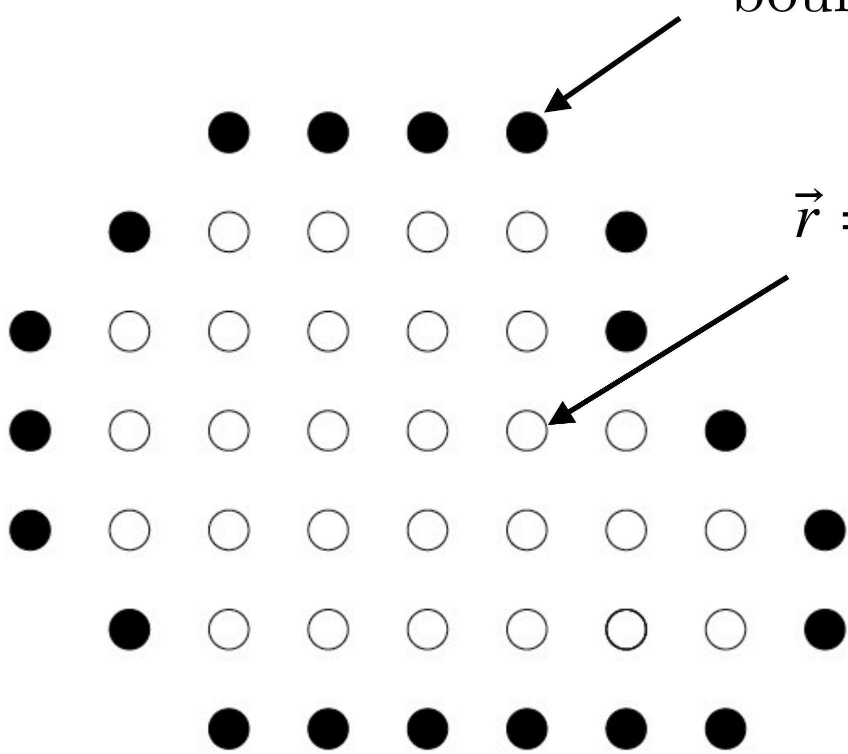


Animating Schrödinger's Equation

https://www.youtube.com/watch?v=Xj9PdeY64rA

Discretization

boundary ∂V



$\overrightarrow{e}_i, i = 1,\cdots, d$ unit vectors of a d-dimensional hyper cubic lattice

$$\vec{r} = \sum_{i=1}^{d=2} h_i r_i \overrightarrow{e}_i = \sum_{i=1}^{d=2} r_i \overrightarrow{e}_i$$

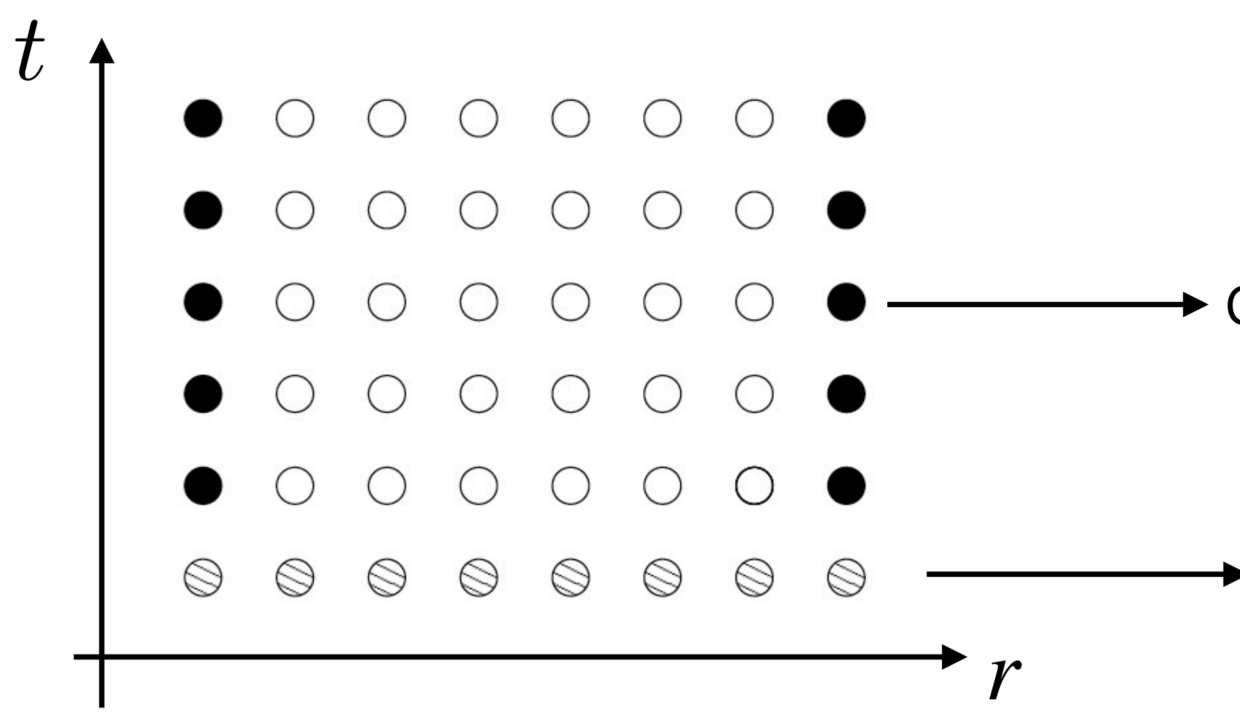
$h_i = h = 1$ lattice spacing

r_i defines the domain V

$$t_n = n\tau, n = 0, 1, 2, \cdots$$

$$u(\vec{r}, t) \rightarrow u(\vec{r}_i, t_n)$$

Discretisation of space-time domain



Compatible boundary conditions

Initial values

$$u(\vec{r}, t = 0), \frac{\partial u(\vec{r}, t)}{\partial t}\Big|_{t=0}$$

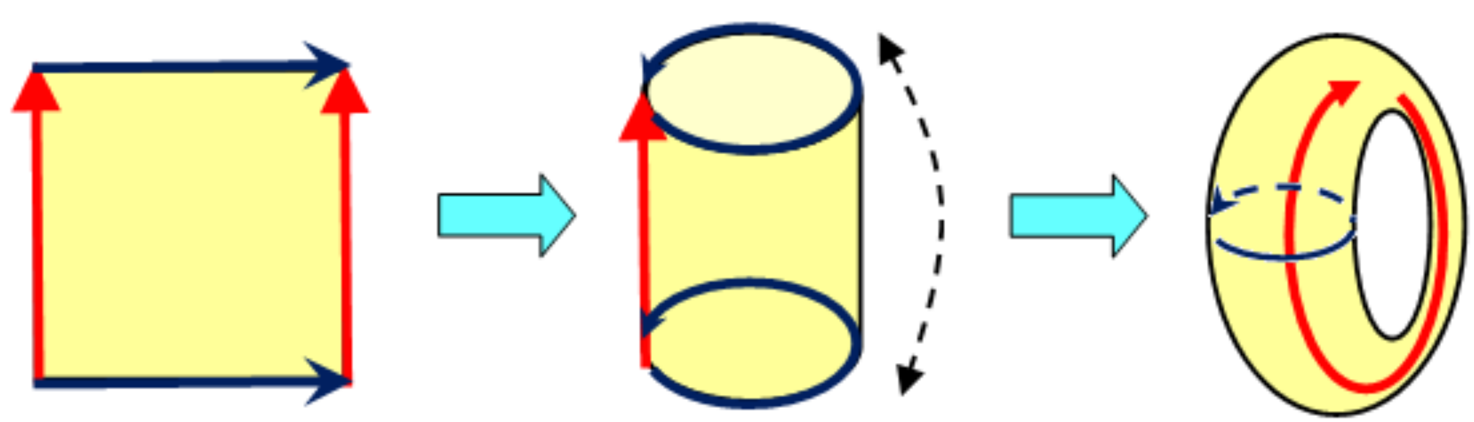
Forward time (FT) discretisation
$$\frac{\partial u(\vec{r}, t_n)}{\partial t} \to \frac{u(\vec{r}, t_{n+1}) - u(\vec{r}, t_n)}{\tau} + O(\tau)$$

Centered space (CS) discretisation
$$\frac{\partial u(\vec{r}, t_n)}{\partial x_i} \to \frac{u(\vec{r} + h_i \vec{e}_i, t_n) - u(\vec{r} - h_i \vec{e}_i, t_n)}{2h_i} + O(h^2)$$

$$\frac{\partial^2 u(\vec{r}, t_n)}{\partial x_i^2} \to \frac{u(\vec{r} + h_i \vec{e}_i, t_n) + u(\vec{r} - h_i \vec{e}_i, t_n) - 2u(\vec{r}, t_n)}{h_i^2} + O(h^2)$$

Boundary conditions

Periodic boundary conditions (PBC) $\quad u(\vec{r} + N_i h_i \vec{e}_i) = u(\vec{r})$



1d ring
2d torus (donut)
higher-d tori

Poisson equation (Potential Problems)

$$\Delta\phi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r}) \qquad \vec{E} = -\nabla\phi \qquad \Delta\phi(\vec{r}) = 4\pi G\rho(\vec{r}) \qquad \vec{g} = -\nabla\phi$$

$$\phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0|\vec{r}|} \qquad \phi(\vec{r}) = -\frac{Gm}{|\vec{r}|}$$

1781-1840
French mathematician, physicist

Centered three-point formula in each direction

$$\Delta\phi(\vec{r}) \to \frac{1}{h^2}\left(-2d\phi(\vec{r}) + \sum_{i=1}^{d}\left(\phi(\vec{r}+h\vec{e}_i) + \phi(\vec{r}-h\vec{e}_i)\right)\right) + O(h^2)$$

$$\phi(\vec{r}) = \frac{1}{2d}\sum_{i=1}^{d}\left(\phi(\vec{r}+h\vec{e}_i) + \phi(\vec{r}-h\vec{e}_i)\right) + \frac{h^2}{2d\epsilon_0}\rho(\vec{r})$$

- Jacobi Relaxation

1. Initial configuration for $\phi(\vec{r})$ (an educated guess) with boundary conditions
2. Calculation a new configuration $\phi^{new}(\vec{r}) = \frac{1}{2d}\sum_{i=1}^{d}[\phi(\vec{r}+h\vec{e}_i) + \phi(\vec{r}-h\vec{e}_i)] + \frac{h^2}{2d\epsilon_0}\rho(\vec{r})$

   Use the boundary condition to determine $\phi^{new}(\vec{r}), \vec{r} \in \partial V$

3. Calculate the deviation $\delta\phi = \max_{\vec{r}} |\phi^{new}(\vec{r}) - \phi(\vec{r})|$
4. Replace ϕ by ϕ^{new}. If $\delta\phi$ smaller than the bound, stop; otherwise, repeat from step 2.

Relax towards the actual solution from boundary to the interior of V

Computational complexity $\quad \#\ \text{iterations} \approx \frac{1}{2} p N_s^{2/d}$ to improve the error by 10^{-p}

$$\text{for } d = 2, \sim N_s$$

Gauss-Seidel Relaxation $\quad$ no need to work with two arrays ϕ and ϕ^{new}

Change the 2nd step in Jacobi

$$\phi(\vec{r}) = \frac{1}{2d} \sum_{i=1}^{d} (\phi(\vec{r} + h\vec{e}_i) + \phi(\vec{r} - h\vec{e}_i)) + \frac{h^2}{2d\epsilon_0} \rho(\vec{r})$$

As one systematically carry out the relaxation

$\phi(\vec{r})$ contains both old and new values during the iteration through the lattice

Save the memory

Speed-up $\quad \#\ \text{iterations} \approx \frac{1}{4} p N_s^{2/d}$

Successive Overrelaxation (SOR)

$$\phi(\vec{r}) \rightarrow (1-\omega)\phi(\vec{r}) + \omega\left(\frac{1}{2d}\sum_{i=1}^{d}(\phi(\vec{r}+h\vec{e}_i) + \phi(\vec{r}-h\vec{e}_i)) + \frac{h^2}{2d\epsilon_0}\rho(\vec{r})\right)$$

relaxation parameter ω Gauss-Seidel $\omega = 1$

Underrelaxation $0 < \omega < 1$ overrelaxation $1 < \omega < 2$

Put more weight to the new value than in Gauss-Seidel

- Stable for $\omega < 2$
- Converges faster than Gauss-Seidel
- Optimal value of the ω_{opt} close to 2
- $\# \text{ iterations } \approx \frac{1}{3} p N_s^{1/d}$

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\frac{1}{4}(\cos(\frac{\pi}{N_x}) + \cos(\frac{\pi}{N_y}))^2}} \approx 1.939$$

with N=100

$N_x \times N_y$ square lattice with Dirichlet boundary condition

$$d = 2, \quad N_s^{2/d} \sim N_s$$

$$d = 2, \quad N_s^{1/d} \sim \sqrt{N_s}$$

Our bible

Chap. 20.5.

http://numerical.recipes/book.html

Matrix-Formulation

$$\Delta\phi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})$$

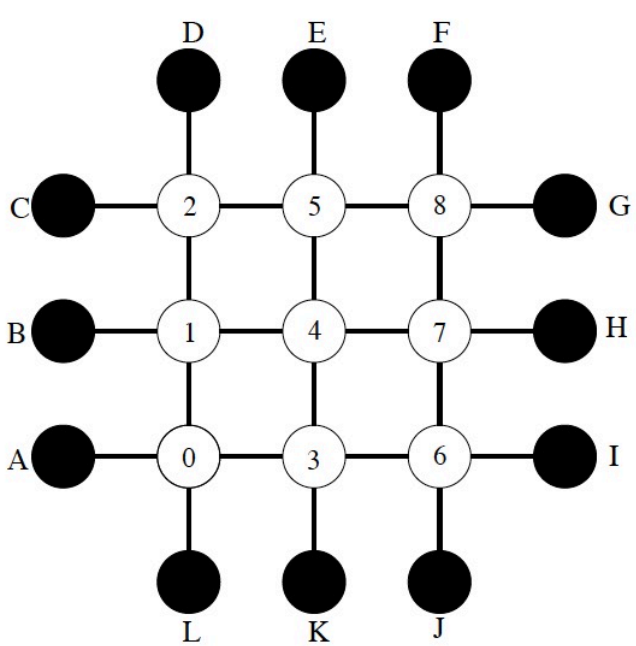
$$\phi(\vec{r}) = \frac{1}{2d}\sum_{i=1}^{d}(\phi(\vec{r}+h\vec{e}_i)+\phi(\vec{r}-h\vec{e}_i)) + \frac{h^2}{2d\epsilon_0}\rho(\vec{r})$$

A set of linear equations $A\vec{\phi} = \vec{b}$

$$\vec{\phi} = \begin{pmatrix} \phi(\vec{x}_0) \\ \phi(\vec{x}_{\vec{r}_1}) \\ \vdots \\ \phi(\vec{x}_{\vec{r}_{N-1}}) \end{pmatrix}$$

$\vec{b}$ boundary condition

A Laplace operator $-\Delta$



$$\text{on site } 0: -\phi_3 - \phi_A - \phi_1 - \phi_L + 4\phi_0 = \frac{h^2}{\epsilon_0}\rho_0$$

$$\text{on site } 1: -\phi_4 - \phi_B - \phi_2 - \phi_0 + 4\phi_1 = \frac{h^2}{\epsilon_0}\rho_1$$

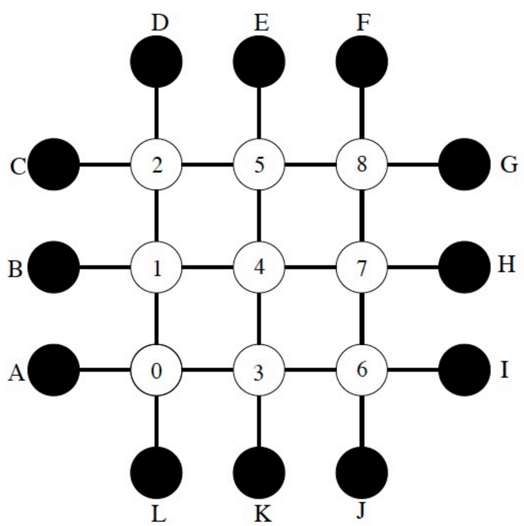
$$\vdots$$

$$\text{on site } 4: -\phi_7 - \phi_1 - \phi_5 - \phi_3 + 4\phi_4 = \frac{h^2}{\epsilon_0}\rho_4$$

$$\vdots$$

$$\text{on site } 8: -\phi_G - \phi_5 - \phi_F - \phi_7 + 4\phi_8 = \frac{h^2}{\epsilon_0}\rho_8$$

Matrix-Formulation



$$
\begin{aligned}
-\phi_3 - \phi_1 + 4\phi_0 &= \frac{h^2}{\epsilon_0}\rho_0 + \phi_A + \phi_L =: b_0 \\
-\phi_4 - \phi_2 - \phi_0 + 4\phi_1 &= \frac{h^2}{\epsilon_0}\rho_1 + \phi_B =: b_1 \\
&\vdots \\
-\phi_7 - \phi_1 - \phi_5 - \phi_3 + 4\phi_4 &= \frac{h^2}{\epsilon_0}\rho_4 =: b_4 \\
&\vdots \\
-\phi_5 - \phi_7 + 4\phi_8 &= \frac{h^2}{\epsilon_0}\rho_8 + \phi_G + \phi_F =: b_8
\end{aligned}
$$

$$A\vec{\phi} = \vec{b}$$

$$
\begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix}
=
\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{pmatrix}
$$

- A is symmetric $A^T = A$
- A is sparse $A_{ii} = 2d, A_{ij} = -1 \text{ only if } \langle i, j\rangle$
- A is positive definite $\vec{x}^T A\vec{x} > 0, \text{ for } \vec{x} \neq 0$

$$\int d^dx f^*(\vec{x})(-\Delta)f(\vec{x}) = \int d^dx |\nabla f(\vec{x})|^2 > 0,$$

Partial integration over an infinite domain

Positive semi-definite $v^T Q v = v^T X^T X v = (Xv)^T (Xv) = u^T u \geq 0$

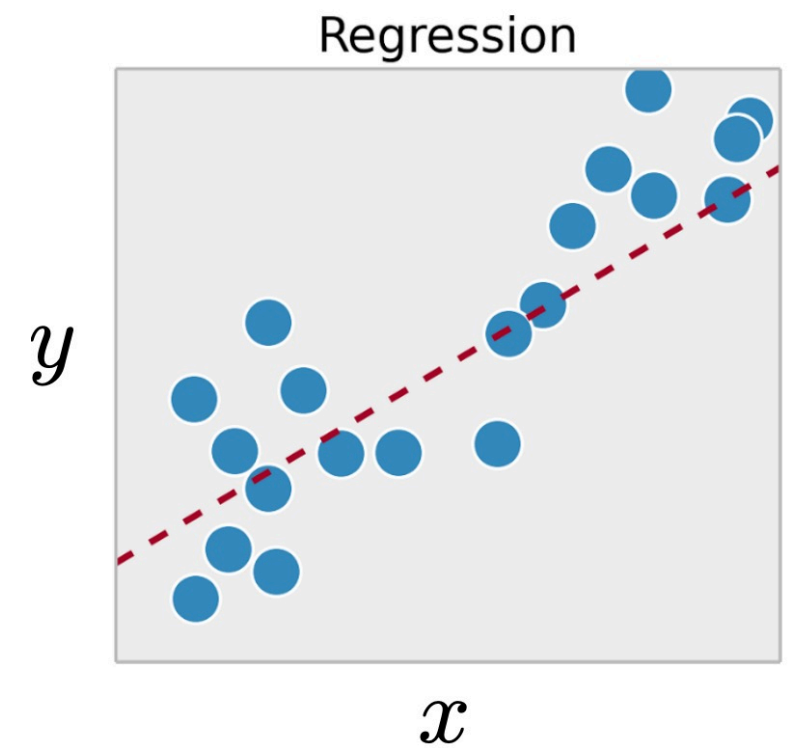
Remember what we learnt in the machine learning lecture ?

https://quantummc.xyz/wp-content/uploads/2023/03/lecture1_1.pdf

$$\{(x_j^{(i)}, y^{(i)}), \theta_j\};\ j = 1, 2, \cdots, N;\ i = 1, 2, \cdots, M; N < M$$

$$y^{(i)} = \theta_0 + \theta_1 x_1^{(i)} + \theta_2 x_2^{(i)} + \cdots + \theta_N x_N^{(i)}$$

$$\mathbb{R}^{M\times(N+1)} \quad \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & \cdots & x_N^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & \cdots & x_N^{(2)} \\ & \vdots & & \vdots & \\ 1 & x_1^{(M)} & x_2^{(M)} & \cdots & x_N^{(M)} \end{bmatrix} \cdot \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_N \end{bmatrix} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(M)} \end{bmatrix}$$



$$\underline{\underline{X}} \cdot \underline{\Theta} = \underline{Y}$$

Paraboloid and positive-definite

$$f(x) = \frac{1}{2} x^T A x - b^T x + c \qquad \nabla_x f(x) = Ax - b = 0 \qquad Ax = b$$

$$A = \frac{1}{M} X^T X \qquad \mathbb{R}^{(N+1)\times(N+1)}$$

$$b = X^T Y \qquad \mathbb{R}^{(N+1)\times 1}$$

$$A = \underbrace{\begin{bmatrix} \langle 1 \rangle & \langle x_1 \rangle & \langle x_2 \rangle & \cdots & \langle x_N \rangle \\ \langle x_1 \rangle & \langle (x_1)^2 \rangle & \langle x_1 x_2 \rangle & \cdots & \langle x_1 x_N \rangle \\ \langle x_2 \rangle & \langle x_2 x_1 \rangle & \langle (x_2)^2 \rangle & \cdots & \langle x_2 x_N \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle x_N \rangle & \langle x_N x_1 \rangle & \langle x_N x_2 \rangle & \cdots & \langle (x_N)^2 \rangle \end{bmatrix}}_{\mathbb{R}^{(N+1)\times(N+1)}}$$

Gauss Elimination

$$\begin{array}{rcrcrcr} x_0 & + & x_1 & + & x_2 & = & 6 \\ -x_0 & + & 2x_1 & & & = & 3 \\ 2x_0 & & & + & x_2 & = & 5 \end{array}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}, \quad \text{and } \vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

Eliminate x0 from 2nd and 3rd equation

$$\begin{array}{rcrcrcr} x_0 & + & x_1 & + & x_2 & = & 6 \\ & & 3x_1 & + & x_2 & = & 9 \\ & & -2x_1 & - & x_2 & = & -7 \end{array}$$

Forward elimination $O(n^3)$

For $j = 0, ..., n-2$ do:
For $i = j+1, ..., n-1$ do:

Test with 3×3 matrix

$$\begin{array}{rcl} f & := & -A_{ij}/A_{jj} \\ A_{ik} & \rightarrow & A_{ik} + f A_{jk}, \quad k = j, ..., n-1 \\ b_i & \rightarrow & b_i + f b_j. \end{array}$$

Eliminate x1 from 3rd equation

$$\begin{array}{rcrcrcr} x_0 & + & x_1 & + & x_2 & = & 6 \\ & & 3x_1 & + & x_2 & = & 9 \\ & & & - & \frac{1}{3}x_2 & = & -1 \end{array}$$

Trigonal system $(\tilde{A}, \vec{\tilde{b}})$

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \quad \vec{\tilde{b}} = \begin{pmatrix} 6 \\ 9 \\ -1 \end{pmatrix}$$

backsubstitution $O(n^2)$

$$x_{n-1} = \frac{\tilde{b}_{n-1}}{\tilde{A}_{n-1,n-1}}, \qquad x_i = \frac{1}{\tilde{A}_{ii}} \left[\tilde{b}_i - \sum_{j=i+1}^{n-1} \tilde{A}_{ij} x_j \right], \quad i = n-2, ..., 0.$$

Problem: after a few iterations, A_{ii} (almost) zero Divide very small numbers, leads to breakdown

Gauss Elimination with pivoting

- Rescale each row of $(A, \vec{b}\,)$ so that the maximum element in each row of A equals 1.
- Initialize the index vector, $v(i) = i$, $i = 0, ..., n-1$. for bookkeeping
- For $j = 0, ..., n-2$ do:
  - Search for the pivot element: $|A_{v(p),j}| = \max\{|A_{v(i),j}|, i = j, ..., n-1\}$
    If $j \neq p$, exchange the j-th and p-th entries of v: $v(j) \leftrightarrow v(p)$
    This corresponds to exchanging the j-th and p-th row.
  - For $i = j+1, ..., n-1$ do:

$$
\begin{aligned}
f &:= -A_{v(i)j} \,/ A_{v(j)j} \\
A_{v(i)k} &\rightarrow A_{v(i)k} + f\, A_{v(j)k}, \quad k = j, ..., n-1 \\
b_{v(i)} &\rightarrow b_{v(i)} + f\, b_{v(j)}.
\end{aligned}
$$

Backsubstitution with trigonal system $(\tilde{A}, \vec{\tilde{b}})$

$$
x_{n-1} = \frac{\tilde{b}_{v(n-1)}}{\tilde{A}_{v(n-1),n-1}}, \qquad x_i = \frac{1}{\tilde{A}_{v(i)i}} \left[\tilde{b}_{v(i)} - \sum_{j=i+1}^{n-1} \tilde{A}_{v(i)j} x_j \right], \quad i = n-2, ..., 0.
$$

LU Decomposition

$$\tilde{A} = MA \text{ where } M = M^{(n-2)} M^{(n-1)} \dots M^{(1)} M^{(0)}$$

$\tilde{A}$ is upper triangular matrix

$M^{(j)}$ are lower triangular matrices

$$A = \underbrace{M^{-1}}_{L} \cdot \underbrace{\tilde{A}}_{U} = L \cdot U$$

$$M_{ii}^{(j)} = 1$$

$$M_{ij}^{(j)} = -\frac{A_{ij}}{A_{jj}}, i > j$$

$$M_{ij}^{(j)} = 0 \text{ otherwise}$$

Complexity still scales $O(N^3)$

$O(N^2)$ once the LU decomposition is known.

$$A\vec{x} = LU\vec{x} = L(U\vec{x}) = \vec{b}$$

Forward substitution $L\vec{y} = \vec{b}$

$$y_0 = \frac{b_0}{L_{00}}, \qquad y_i = \frac{1}{L_{ii}} \left[b_i - \sum_{j=0}^{i-1} L_{ij} y_j \right], \quad i = 1, ..., n-1$$

Backsubstitution $U\vec{x} = \vec{y}$

$$x_{n-1} = \frac{y_{n-1}}{U_{n-1,n-1}}, \qquad x_i = \frac{1}{U_{ii}} \left[y_i - \sum_{j=i+1}^{n-1} U_{ij} x_j \right], \quad i = n-2, ..., 0$$

Our bible

Chap.2.3.

http://numerical.recipes/book/book.html

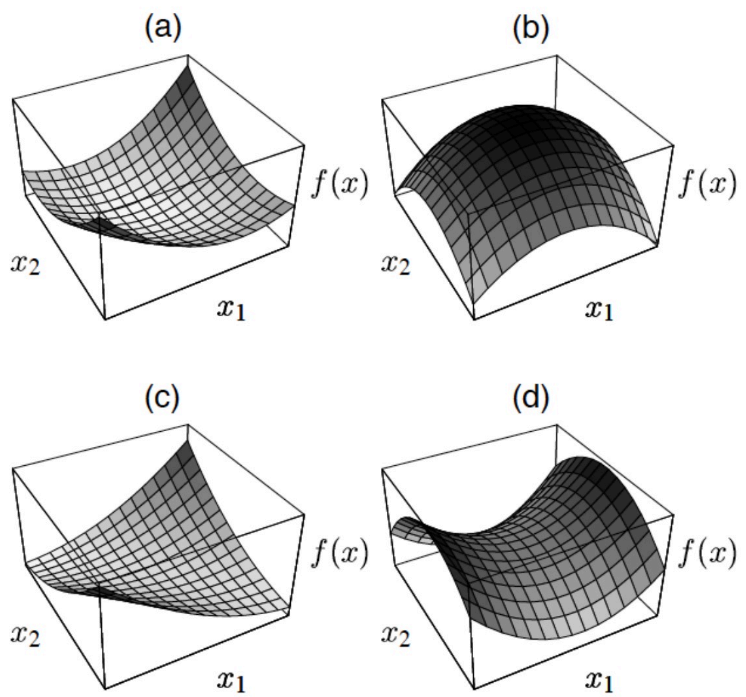
Steepest descent and Conjugate Gradient methods

$$f(x) = \frac{1}{2}x^T A x - b^T x + c \qquad \nabla_x f(x) = Ax - b = 0 \qquad Ax = b$$

$$A = \frac{1}{M}X^T X \quad \mathbb{R}^{(N+1)\times(N+1)}$$

$$b = X^T Y \quad \mathbb{R}^{(N+1)\times 1}$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix} \quad c = 0$$

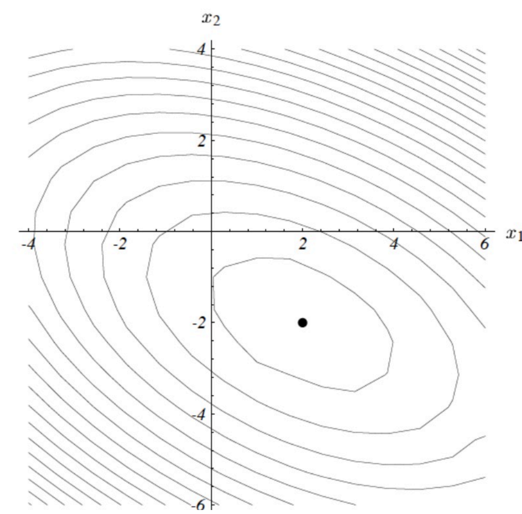


(a) positive-definite, symmetric

(b) negative-definite

(c) singular, a set of solution

(d) saddle point



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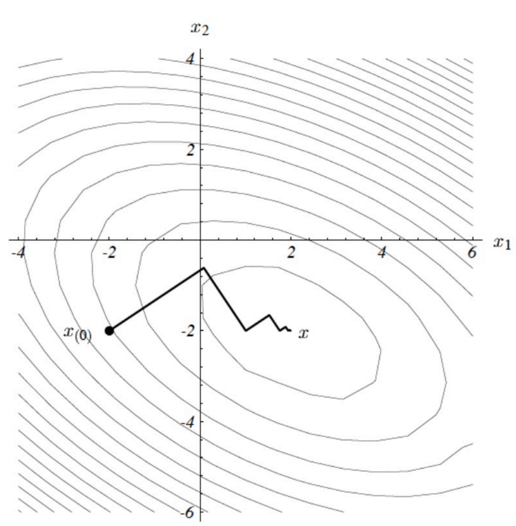
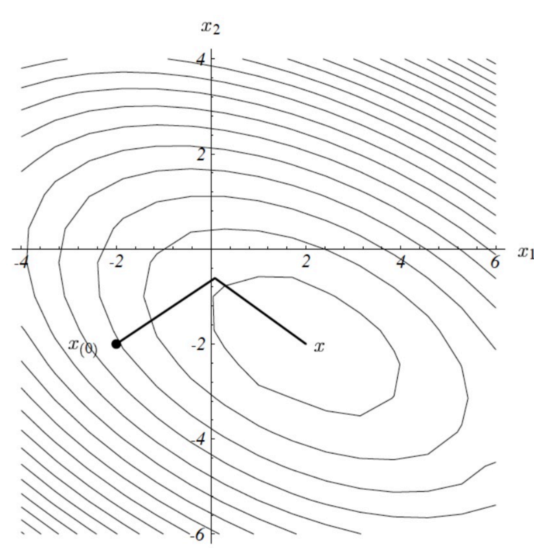
https://quantummc.xyz/wp-content/uploads/2023/03/lecture1_1.pdf

Iteration times $O(N)$



Many steps to find the solution



Only takes N-steps to find the solution

Our bible

Chap.10.8.

http://numerical.recipes/book/book.html

Parabolic PDEs

Time-dependent PDEs: diffusion equation

$$\frac{\partial u(t,\vec{r})}{\partial t} - D\Delta u(t,\vec{r}) = S(t,\vec{r})$$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

$$t_n = n\cdot\tau, \quad n = 0,1,\cdots$$

$$\vec{r} = r\cdot h, \quad r = 0,1,\cdots,N-1$$

FTCS: Forward time (FT) Centered space (CS) scheme

$$\frac{u(n+1,r)-u(n,r)}{\tau} = D\frac{u(n,r+1)-2u(n,r)+u(n,r-1)}{h^2}$$

Also called Explicit Euler method

$$u(n+1,r) = u(n,r) + \frac{D\tau}{h^2}(u(n,r+1)-2u(n,r)+u(n,r-1))$$

Gaussian wave package as a solution

(can you see this ?)

$$u(t,r) = \frac{1}{\sqrt{2\pi}\sigma(t)}\exp[-\frac{r^2}{2\sigma(t)^2}] \qquad \sigma(t) = \sqrt{2Dt}$$

Characteristic diffusion-time $t_h = \frac{h^2}{2D}$

Width σ Increase from 0 to h

Stability $\tau << t_h$

Von Neumann Stability Analysis

Plane wave ansatz $u(n,r) = A^n e^{ikrh}$ $\quad u(n+1,r) = u(n,r) + \frac{D\tau}{h^2}(u(n,r+1) - 2u(n,r) + u(n,r-1))$

$$AA^n e^{ikrh} = A^n e^{ikrh} + \frac{D\tau}{h^2}(A^n e^{ikrh}e^{ikh} - 2A^n e^{ikrh} + A^n e^{ikrh}e^{-ikh})$$

$$A = 1 + \frac{D\tau}{h^2}(e^{ikh} + e^{-ikh} - 2)$$

$$= 1 - 2\frac{D\tau}{h^2}(1 - \cos(kh))$$

$$= 1 - \frac{4D\tau}{h^2}\sin^2(\frac{kh}{2})$$

decrease h to h/2
decrease tau to tau/4

$d = 1$ $\quad |A| \leq 1 \longrightarrow \frac{4D\tau}{h^2} \leq 2 \longrightarrow \tau \leq \frac{h^2}{2D}$ $\quad$ Higher dimension d $\quad \tau \leq \frac{h^2}{2dD}$

With maximum $\tau = \frac{h^2}{2D}$ FTCS becomes $u(n+1,r) = \frac{1}{2}(u(n,r+1) + u(n,r-1))$

Similar with the Jacobi relaxation for Poisson equation when $d = 1$ and $\rho = 0$

https://en.wikipedia.org/wiki/Von_Neumann_stability_analysis

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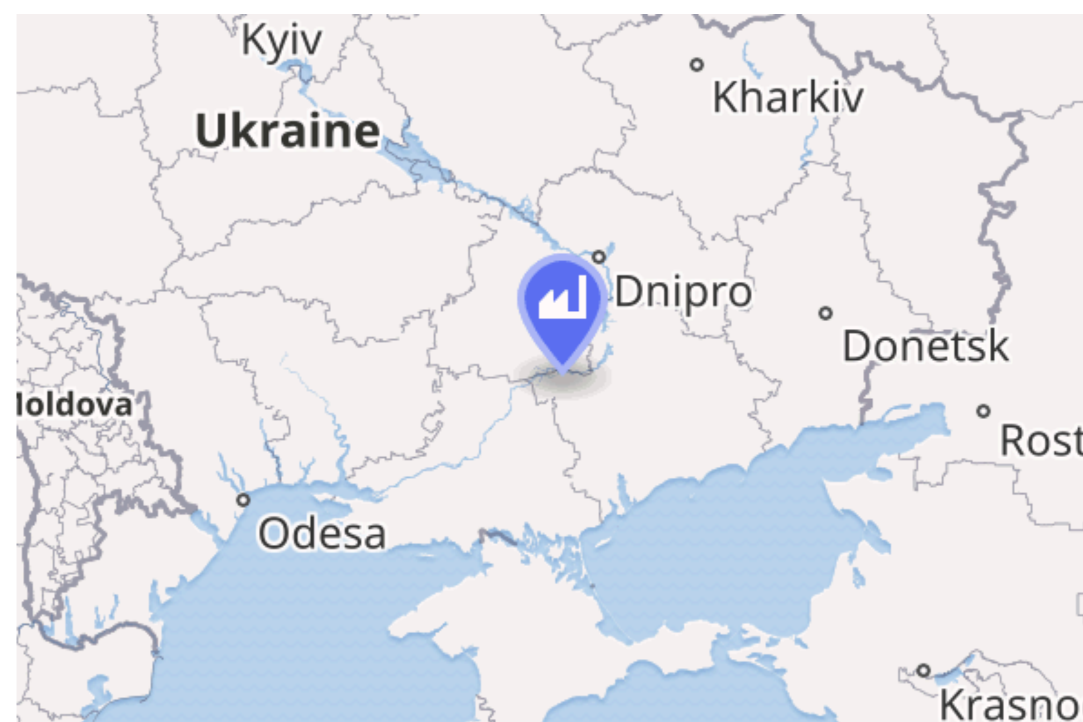
Ukraine: Zaporizhzhya nuclear plant initiates reactor shutdown following water leak, reports IAEA

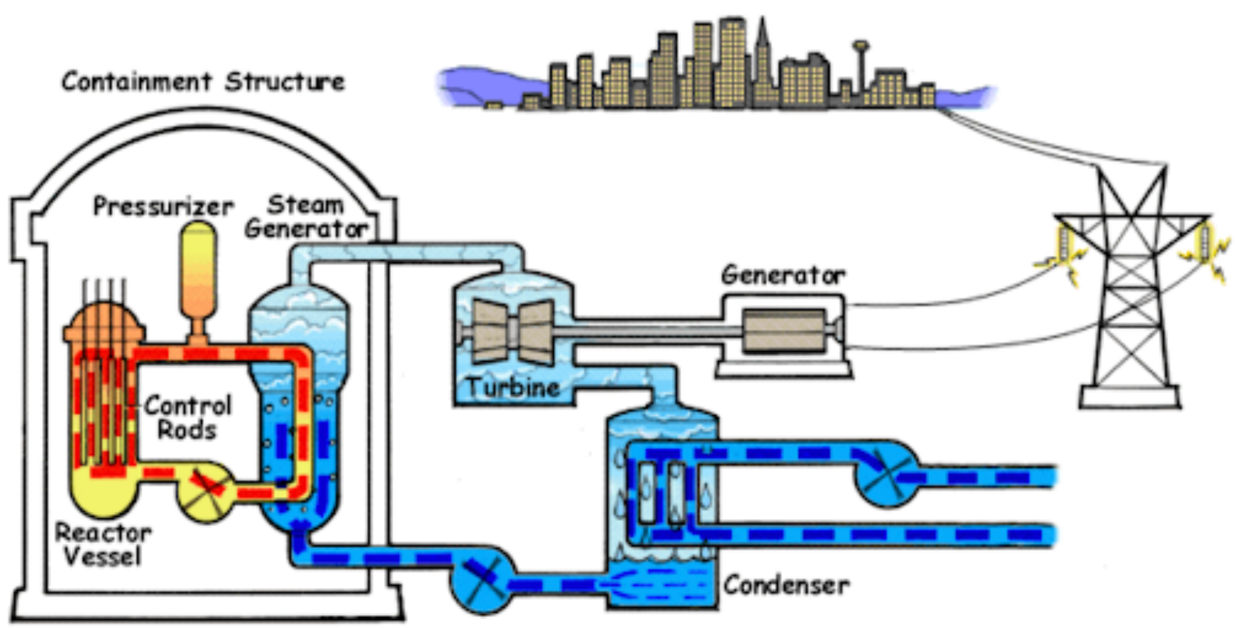
© IAEA/Fredrik Dahl | An IAEA expert mission team walks around the Zaporizhzhya Nuclear Power Plant and its surrounding area. (file)

10 August 2023 | Peace and Security

The **Zaporizhzhia Nuclear Power Station** (Ukrainian: Запорізька атомна електростанція, romanized: *Zaporiz'ka atomna elektrostantsiia*) in southeastern Ukraine is the largest nuclear power plant in Europe and among the 10 largest in the world.



Diffusion of neutrons from the chain reaction of Uranium



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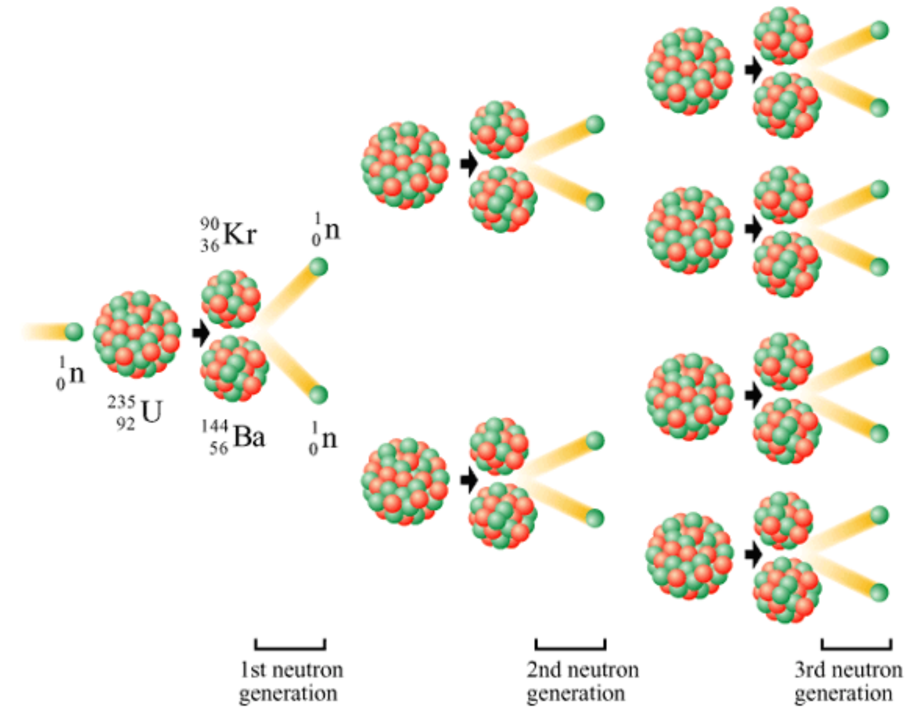

$$^{235}U + n \rightarrow {}^{139}Ba + {}^{95}Kr + 2n$$

For ^{235}U Diffusion constant $D \approx 10^5 m^2/s$ Creation rate $C \approx 10^8 s^{-1}$

$$\frac{\partial}{\partial t} n(t, \vec{r}) = D \Delta n(t, \vec{r}) + C n(t, \vec{r})$$

Dirichlet BC $n(\partial V) = 0 \qquad V = L^d$

Critical length $L_c = \pi \sqrt{\frac{dD}{C}}$

FTCS $n(n+1, r) = (1 + C\tau) n(n, r)$

$$+ \frac{D\tau}{h^2}(u(n, r+1) - 2u(n, r) + u(n, r-1))$$

Stability $\tau \leq \frac{h^2}{2dD} = \frac{h^2}{4D} \qquad d = 2$

$$D = C = 1, L_c = \sqrt{2}\pi$$