
# Content

**0. Introduction**

**1. Regression**

**1.1 Multivariate Linear Regression (curve fitting)**

**1.2 Regularization (Lagrange multiplier)**

**1.3 Logistic Regression (Fermi-Dirac distribution)**

**1.4 Support Vector Machine (high-school geometry)**

**2. Dimensionality Reduction/feature extraction**

**2.1 Principal Component Analysis (order parameters)**

**2.2 Recommender Systems**

**2.3 Clustering (phase transition)**

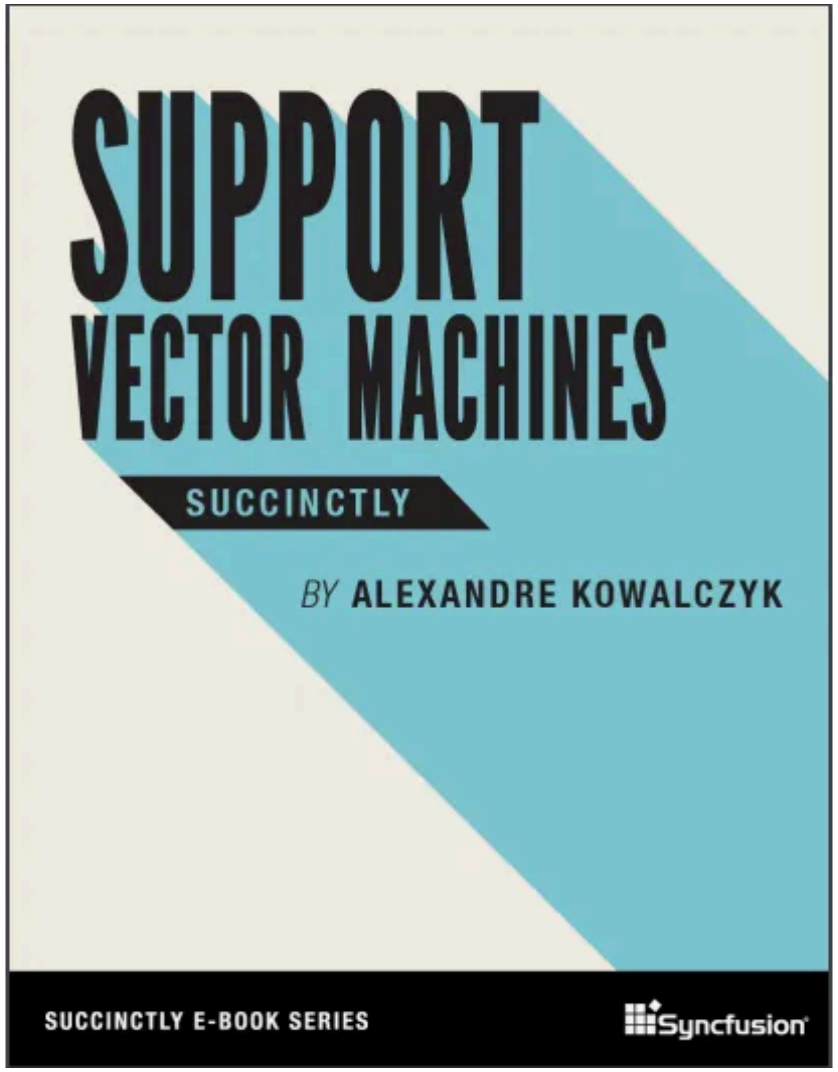
# Content

# Support Vector Machine

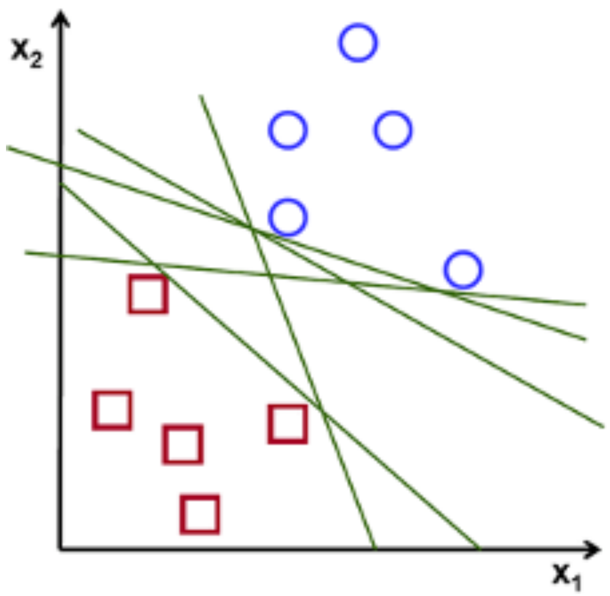
Good references by Alexandre Kowalczyk

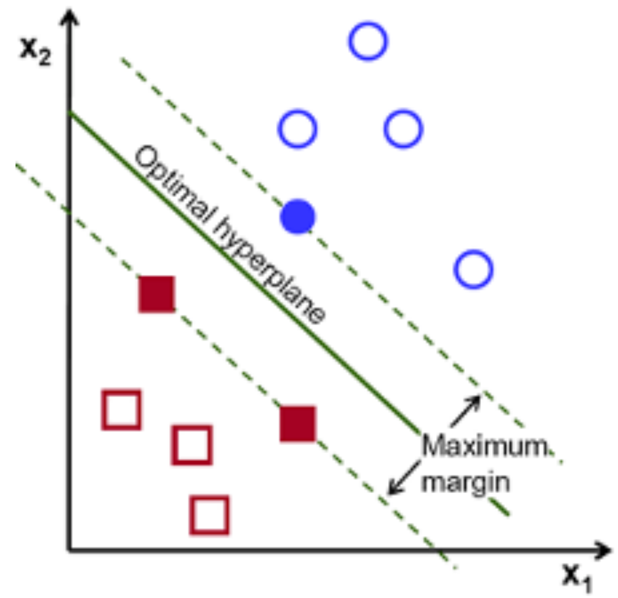
https://www.svm-tutorial.com/

https://www.syncfusion.com/ebooks/support_vector_machines_succinctly



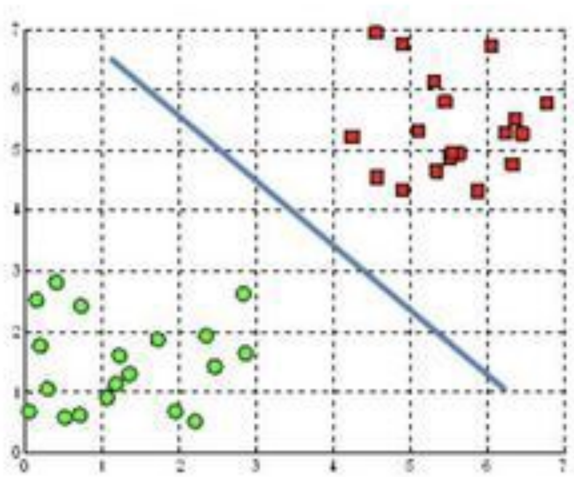
# Support Vector Machine

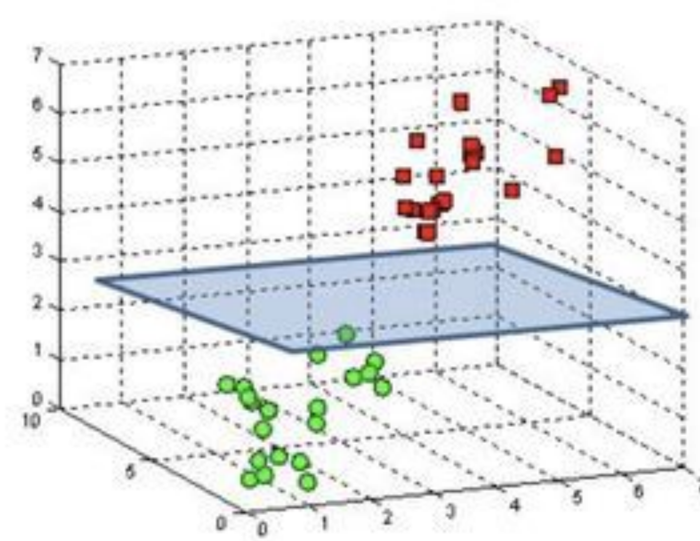


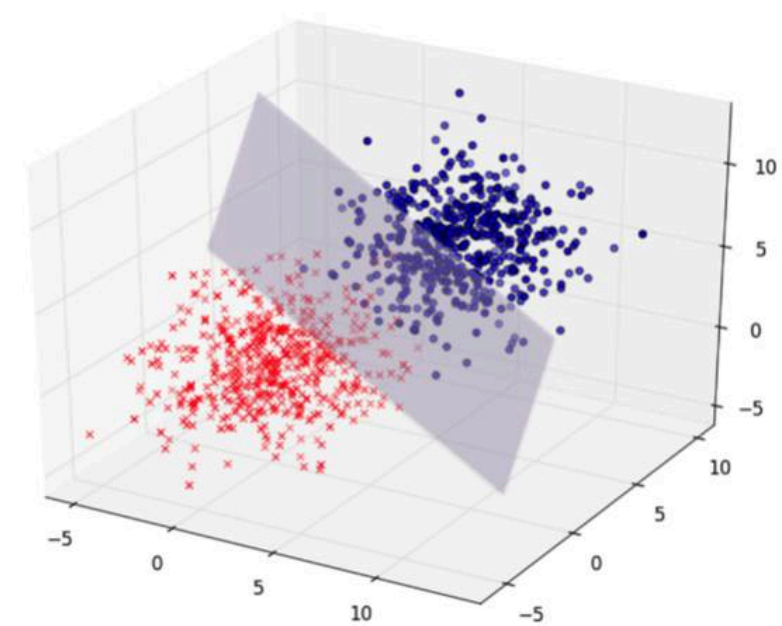




A data point: p-dimensional vector (p=2)
Can be separated by (p-1)-dimensional hyperplane

Hyperplane with maximal margin (largest separation between classes)



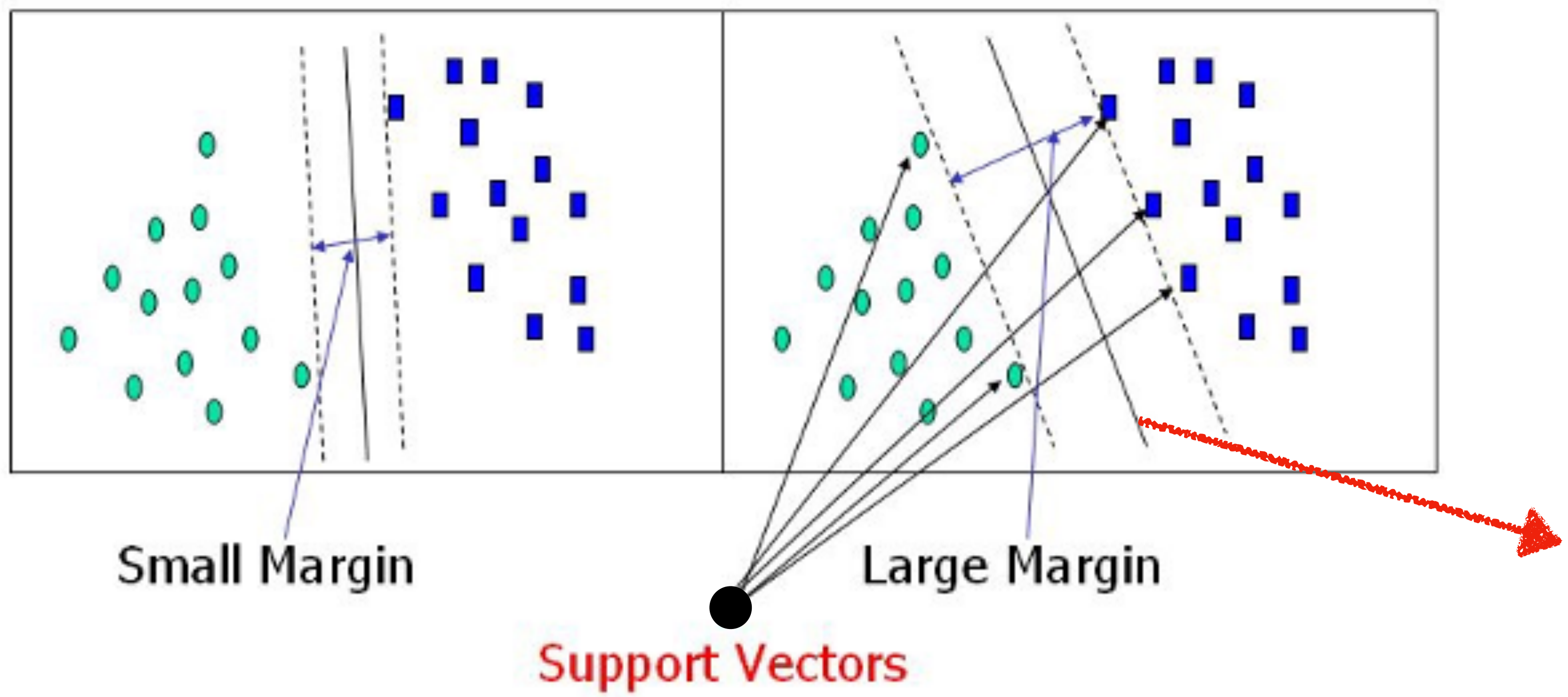




Hyperplane in R^2 is a line

Hyperplane in R^3 is a plane

# Support Vector Machine




$$\vec{\omega} \cdot \vec{x}_i + b = 0$$
$$\theta^T \cdot x = 0$$

## Optimisation problem with constraints:

Linearly separable training set $\mathcal{D} = \{(\vec{x}_i, y_i) | \vec{x}_i \in \mathbb{R}^n, y_i \in \{-1, 1\}\}_{i=1}^{m}$

Geometric margin $M = \min_{i=1,2,\cdots,m} \frac{|y_i(\vec{\omega} \cdot \vec{x}_i + b)|}{||\vec{\omega}||}$

The optimal separating hyperplane is the hyperplane $(\vec{\omega}, b)$ whose margin $M$ is the largest

# Some high-school Geometry

Given a plane $w_1 x_1 + w_2 x_2 + b = 0$

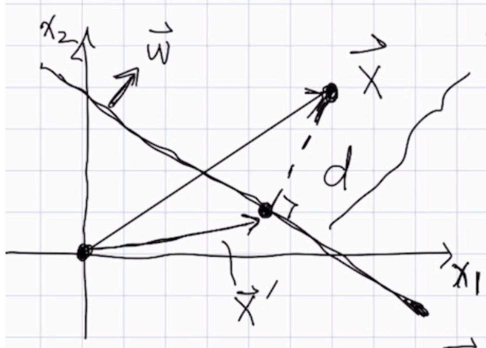
* distance from origin $(0,0)$ to the plane is $\boxed{\dfrac{b}{\sqrt{w_1^2 + w_2^2}} = \dfrac{b}{\|\vec{w}\|}}$

* distance from arbitrary point $(x_1, x_2)$ to the plane is $\boxed{\dfrac{|w_1 x_1 + w_2 x_2 + b|}{\|\vec{w}\|}}$

* $\vec{w} = (w_1, w_2)$ is the normal vector to the plane

$\|\vec{w}\| = \sqrt{w_1^2 + w_2^2}$

Proof of distance from $\vec{x} = (x_1, x_2)$ to the plane $\vec{w} \cdot \vec{x} + b = 0$ $\quad d = \dfrac{\vec{w} \cdot \vec{x} + b}{\|\vec{w}\|}$




$\vec{w} \cdot \vec{x} + b = 0$

$\vec{X} = \vec{X}' + \hat{d}$

$\hat{d} = \dfrac{\vec{w}}{\|\vec{w}\|} d$

$\vec{w} \cdot \vec{x}' + b = 0$

$\vec{w} \cdot (\vec{X} - \hat{d}) + b = 0$

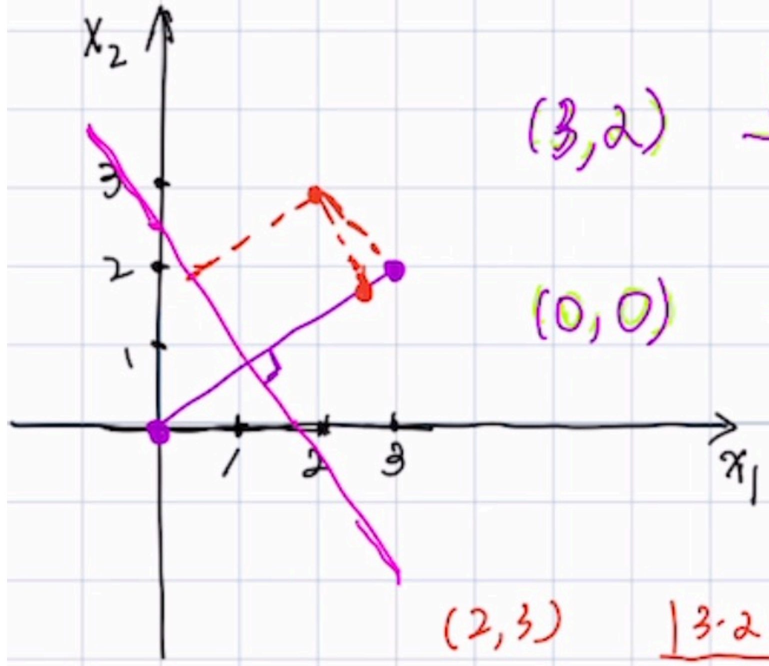
$\vec{w} \cdot \vec{X} - \vec{w} \cdot \dfrac{\vec{w}}{\|\vec{w}\|} d + b = 0$

$d = \dfrac{\vec{w} \cdot \vec{X} + b}{\|\vec{w}\|}$

Example

$3x_1 + 2x_2 - 5 = 0$ $\quad$ normal vector $\vec{w} = (3, 2)$

$\|\vec{w}\| = \sqrt{13}$




$(3,2)$ $\quad \dfrac{|3 \cdot 3 + 2 \cdot 2 - 5|}{\sqrt{13}} = \dfrac{8}{\sqrt{13}}$

$(0,0)$ $\quad \dfrac{|-5|}{\sqrt{13}} = \dfrac{5}{\sqrt{13}}$

$\dfrac{8}{\sqrt{13}} + \dfrac{5}{\sqrt{13}} = \sqrt{13} = \|\vec{w}\|$

$(2,3)$ $\quad \dfrac{|3 \cdot 2 + 2 \cdot 3 - 5|}{\sqrt{13}} = \dfrac{7}{\sqrt{13}}$

distance between $(2,3)$ and $(3,2)$ $\quad \sqrt{1+1} = \sqrt{2}$

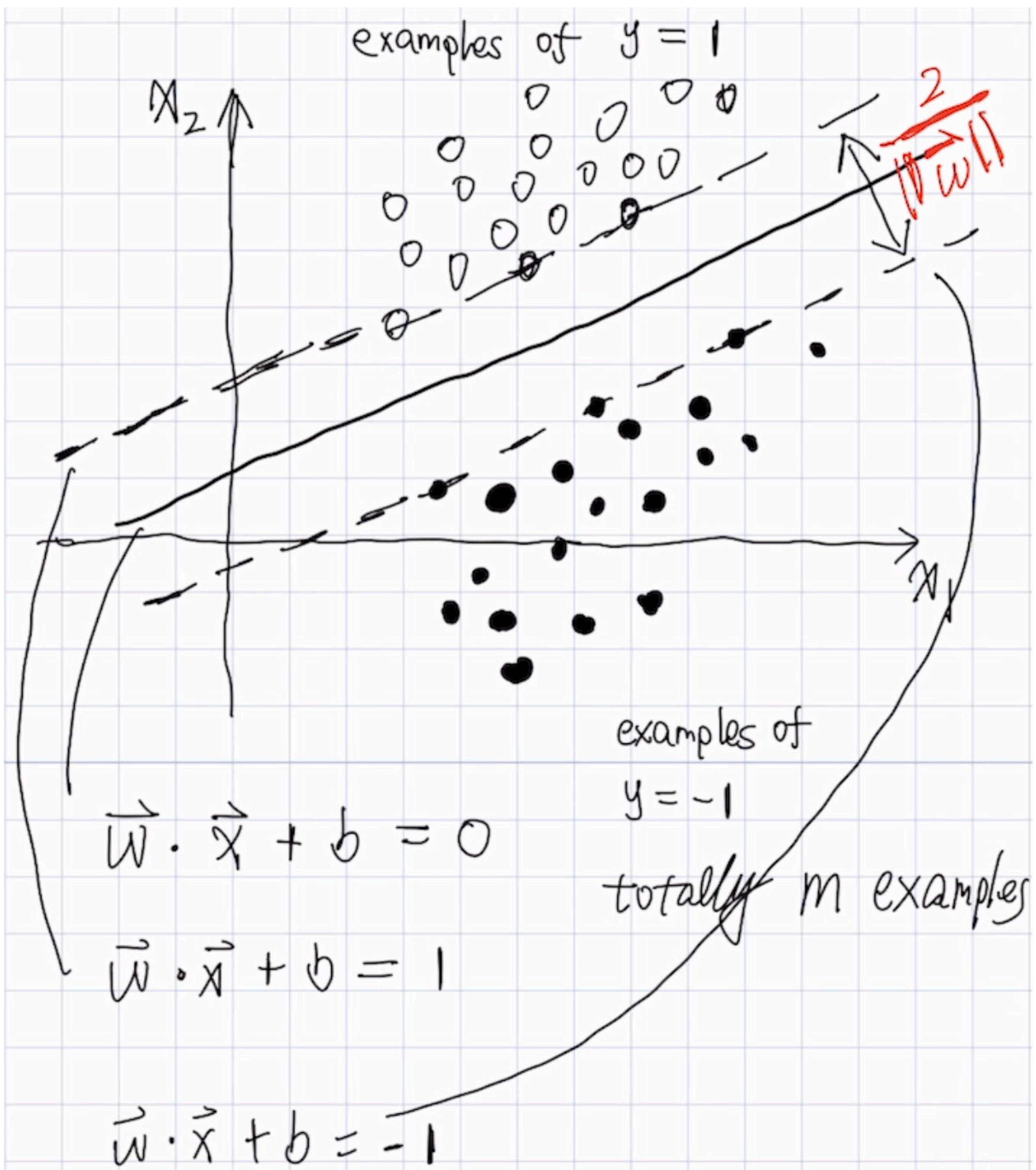
distance between $(2,3)$ and $2x_1 - 3x_2 = 0$ is

$\dfrac{|2 \cdot 2 - 3 \cdot 3|}{\sqrt{13}} = \dfrac{5}{\sqrt{13}}$

so $\quad \sqrt{2} = \sqrt{\left(\dfrac{5}{\sqrt{13}}\right)^2 + \left(\dfrac{1}{\sqrt{13}}\right)^2} = \sqrt{2}$

# Support Vector Machine

How to find the optimal hyperplane for a dataset among all possible hyperplanes




Geometric margin M: distance/2 between two hyperplane

$$M = \frac{1}{||\vec{\omega}||}$$

Maximize the geometric margin means minimize

$$||\vec{\omega}||$$

Constraints: at the same time, prevent data points from falling into the margin

To find $(\vec{\omega}, b)$ such that

$$\max_{(\vec{\omega}, b)} M$$

$$\text{subject to} \quad \frac{|y_i(\vec{\omega} \cdot \vec{x}_i + b)|}{||\vec{\omega}||} \geq M, \quad i = 1, 2, \cdots, m$$

Constrained optimization problem

# Support Vector Machine

To find $(\vec{\omega}, b)$ such that

$$\max_{(\vec{\omega},b)} M$$

$$\text{subject to} \quad \frac{|y_i(\vec{\omega}\cdot\vec{x}_i + b)|}{||\vec{\omega}||} \geq M, \quad i = 1, 2, \cdots, m$$

Is equivalent to the minimisation problem with constraints, remember $M = \dfrac{1}{||\vec{\omega}||}$

$$\min_{(\vec{\omega},b)} ||\vec{\omega}||$$

$$\text{subject to} \quad y_i(\vec{\omega}\cdot\vec{x}_i + b) \geq 1, \; i = 1, 2, \cdots, m$$

Is equivalent to

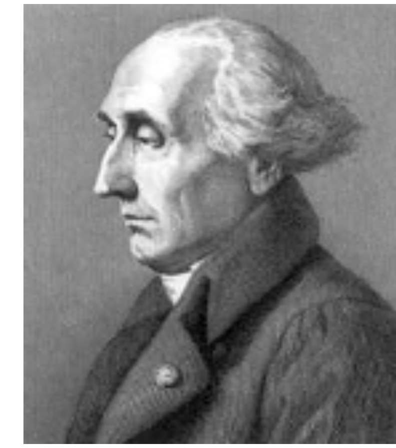
$$\min_{(\vec{\omega},b)} \frac{1}{2}||\vec{\omega}||^2$$

$$\text{subject to} \quad y_i(\vec{\omega}\cdot\vec{x}_i + b) \geq 1, \; i = 1, 2, \cdots, m$$

Lagrange multipliers and duality


Convex quadratic optimisation problem

# Lagrange multipliers and duality



Joseph-Louis Lagrange (1736-1813)

$$\underset{\vec{x}}{\text{minimize}} \quad f(\vec{x})$$
$$\underbrace{\text{subject to} \quad g(\vec{x}) = 0}_{\text{equality constraint}}$$

$$\mathcal{L} = f(\vec{x}) - \alpha g(\vec{x})$$

find the local maxima and minima

$$\nabla \mathcal{L}(\vec{x}, \alpha) = \nabla f(\vec{x}) - \alpha \nabla g(\vec{x}) = 0$$

"I will deduce the complete mechanics of solid and fluid bodies using the principle of least action."

**JOSEPH-LOUIS LAGRANGE**
Letter to Leonhard Euler, May 1756

"I have almost completed a book on analytical mechanics founded solely on the principle [of virtual work]. But since I still have no idea where and when it can be published, I am not in any hurry to finish it."

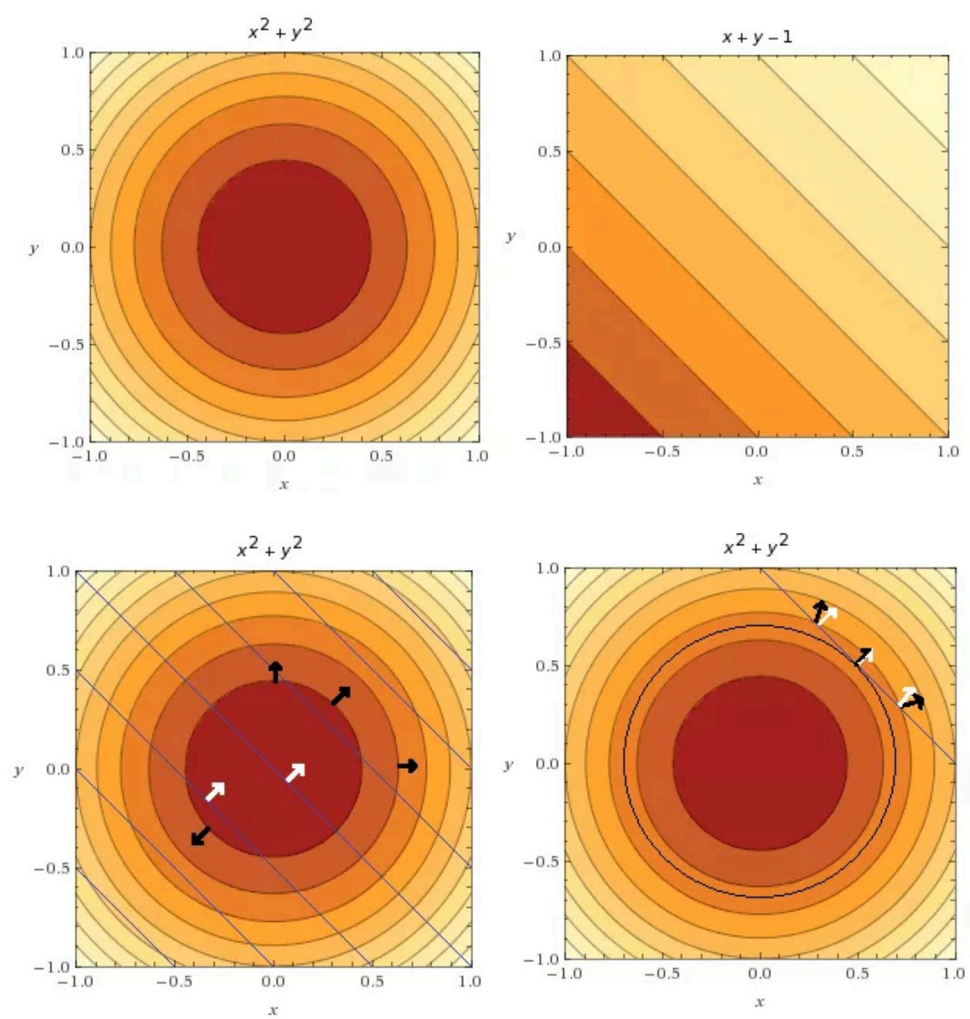
**JOSEPH-LOUIS LAGRANGE**
Letter to Pierre Laplace, September 1782

$$\underset{x,y}{\text{minimize}} \quad f(x,y) = x^2 + y^2$$
$$\text{subject to} \quad g(x,y) = x + y - 1 = 0$$

$$\mathcal{L}(x,y,\alpha) = f(x,y) - \alpha g(x,y)$$

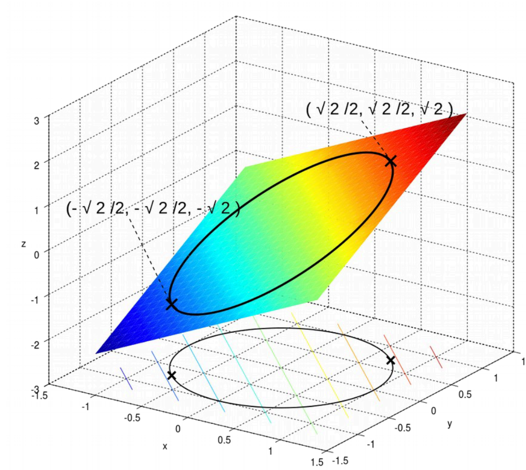
$$\nabla \mathcal{L}(x,y,\alpha) = \nabla f(x,y) - \alpha \nabla g(x,y) = 0$$

$$\nabla_{x_1, x_2, \cdots, x_n, \alpha} \mathcal{L}(x_1, x_2, \cdots, x_n, \alpha) = 0$$




$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \alpha} = 0$$

$$x = y = \frac{1}{2} \qquad \alpha = 1$$




$$f(x,y) = x + y$$

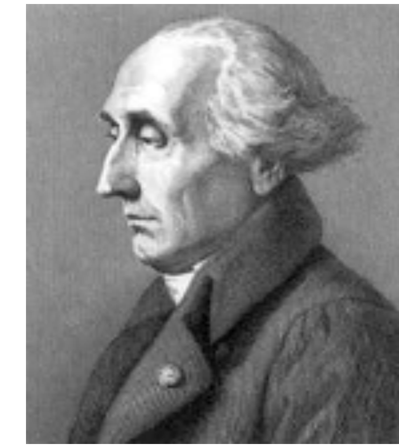
$$g(x,y) = x^2 + y^2 - 1 = 0$$

# Lagrange multipliers and duality

$$\underset{\vec{x}}{\text{minimize}} \quad f(\vec{x})$$
$$\underbrace{\text{subject to} \quad g(\vec{x}) = 0}_{\text{equality constraint}}$$

$$\mathcal{L} = f(\vec{x}) - \alpha g(\vec{x})$$

$$\nabla \mathcal{L}(\vec{x}, \alpha) = \nabla f(\vec{x}) - \alpha \nabla g(\vec{x}) = 0$$



Joseph-Louis Lagrange (1736-1813)

$$L = T - V$$

Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

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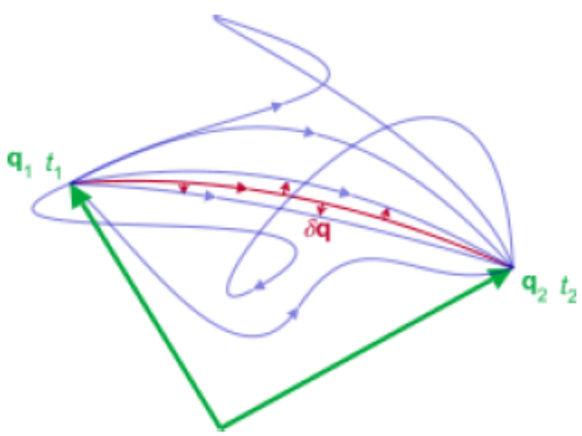
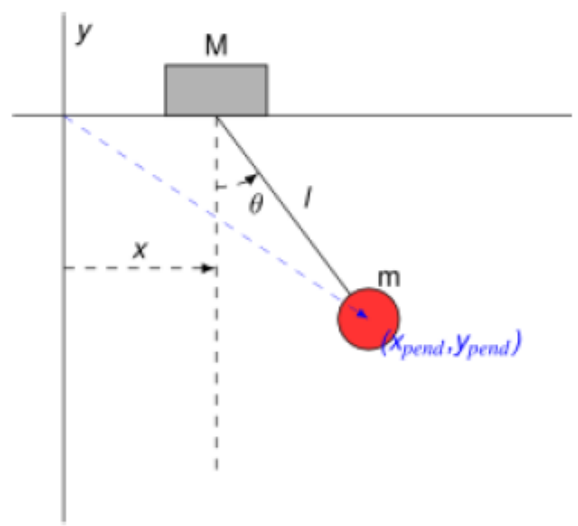



As the system evolves, **q** traces a path through configuration space (only some are shown). The path taken by the system (red) has a stationary action (δS = 0) under small changes in the configuration of the system (δ**q**).[27]




Sketch of the situation with definition of the coordinates (click to enlarge)

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_{\text{pend}}^2 + \dot{y}_{\text{pend}}^2\right)$$

$$V = mgy_{\text{pend}}$$

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[\left(\dot{x} + \ell\dot{\theta}\cos\theta\right)^2 + \left(\ell\dot{\theta}\sin\theta\right)^2\right] + mg\ell\cos\theta \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m(\dot{x}\ell\cos\theta + \ell^2\dot{\theta})\right] + m\ell(\dot{x}\dot{\theta} + g)\sin\theta = 0$$

$$\ddot{\theta} + \frac{\ddot{x}}{\ell}\cos\theta + \frac{g}{\ell}\sin\theta = 0$$

$$S = \int_{t_1}^{t_2} L\,\mathrm{d}t\,, \qquad \delta S = 0$$

*principle of least action*

# Lagrange multipliers and duality

$$\min_{(\vec{\omega},b)} \frac{1}{2}||\vec{\omega}||^2$$

$$\text{subject to} \quad y_i(\vec{\omega}\cdot\vec{x}_i + b) \geq 1,\ i = 1, 2, \cdots, m$$

$$\mathscr{L}(\overrightarrow{\omega}, b, \alpha) = \frac{1}{2}||\overrightarrow{\omega}||^2 - \sum_{i=1}^{m} \alpha_i [y_i(\overrightarrow{\omega}\cdot\overrightarrow{x}_i + b) - 1]$$

$$\min_{(\vec{\omega},b)} \max_{\alpha} \mathcal{L}(\vec{\omega}, b, \alpha)$$

$$\text{subject to} \quad \alpha_i \geq 0, i = 1, 2, \cdots, m$$

$$\nabla_{\vec{\omega}}\mathcal{L} = \vec{\omega} - \sum_{i=1}^{m} \alpha_i y_i \vec{x}_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial b} = -\sum_{i=1}^{m} \alpha_i y_i = 0$$

$$\mathcal{L}_D = \sum_{i=1}^{m} \alpha_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i\alpha_j y_i y_j \vec{x}_i\cdot\vec{x}_j$$

$$\max_{\alpha} \mathcal{L}_D(\alpha, \vec{x}_i, y_i)$$

$$\text{subject to} \quad \alpha_i \geq 0, i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0$$

gradient, not the -gradient

## Karush-Kuhn-Tucker (KKT) conditions

- Stationarity condition:

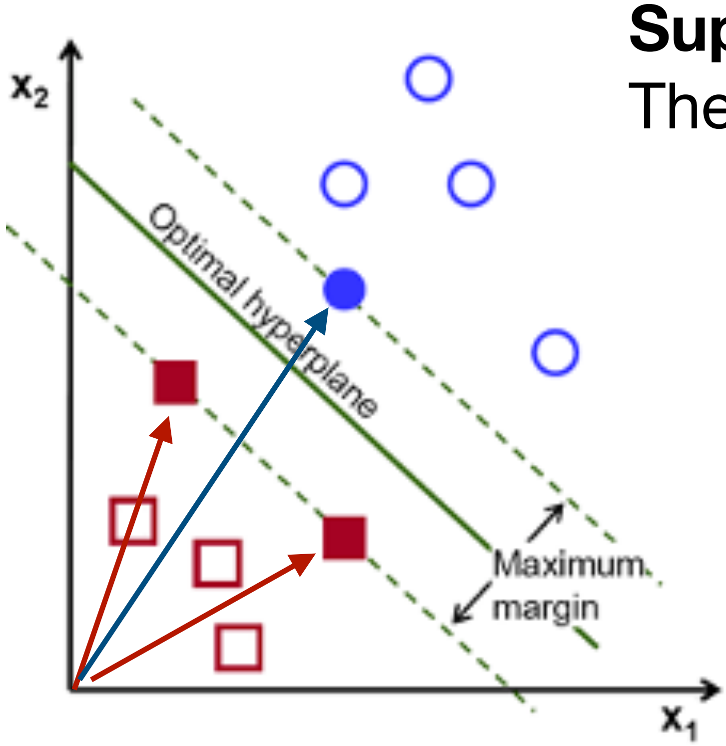
$$\nabla_{\vec{\omega}} \mathcal{L} = \vec{\omega} - \sum_{i=1}^{m} \alpha_i y_i \vec{x}_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial b} = -\sum_{i=1}^{m} \alpha_i y_i = 0$$

- Primal feasibility condition: $y_i(\vec{\omega} \cdot \vec{x}_i + b) - 1 \geq 0 \quad \text{for all } i = 1, 2, \cdots, m$

- Dual feasibility condition: $\alpha_i \geq 0 \quad \text{for all } i = 1, 2, \cdots, m$

- Complementary slackness condition: $\alpha_i [y_i(\vec{\omega} \cdot \vec{x}_i + b) - 1] = 0 \quad \text{for all } i = 1, 2, \cdots, m$




**Support vectors** are examples having a positive Lagrange multiplier. They are the ones the constraint is active.

Once have the multipliers and support vectors

$$\vec{\omega} = \sum_{i=1}^{m} \alpha_i y_i \vec{x}_i$$

$$b = \frac{1}{S} \sum_{i=1}^{S} (y_i - \vec{\omega} \cdot \vec{x}_i)$$

$S$ is the number of support vectors

## Support Vector Machine: Hinge Loss

$$\mathscr{L}(\overrightarrow{\omega}, b, \alpha) = \frac{1}{2}||\overrightarrow{\omega}||^2 - \sum_{i=1}^{m} \alpha_i[y_i(\overrightarrow{\omega} \cdot \overrightarrow{x}_i + b) - 1]$$

$$J(\overrightarrow{\omega}, b) = \frac{1}{2m}||\overrightarrow{\omega}||^2 + \frac{1}{m}\sum_{i=1}^{m} \max(0, 1 - y_i(\overrightarrow{\omega} \cdot \overrightarrow{x}_i + b))$$

Replace $\frac{1}{2m}$ with $\frac{\lambda}{2m}$ for regularisation

$y_i = 1$

$\vec{\omega} \cdot \vec{x}_i + b > 1$ $\quad \nabla_{\overrightarrow{\omega}} J = \frac{1}{m}\overrightarrow{\omega}$

$\vec{\omega} \cdot \vec{x}_i + b < 1$ $\quad \nabla_{\overrightarrow{\omega}} J = \frac{1}{m}\overrightarrow{\omega} - y_i\overrightarrow{x}_i$

$\nabla_b J = -y_i$

$y_i = -1$

$\vec{\omega} \cdot \vec{x}_i + b < -1$ $\quad \nabla_{\overrightarrow{\omega}} J = \frac{1}{m}\overrightarrow{\omega}$

$\vec{\omega} \cdot \vec{x}_i + b > -1$ $\quad \nabla_{\overrightarrow{\omega}} J = \frac{1}{m}\overrightarrow{\omega} - y_i\overrightarrow{x}_i$

$\nabla_b J = -y_i$

### Gradient Descent

$$\vec{\omega} = \vec{\omega} - \alpha\nabla_{\vec{\omega}} J$$

$$b = b - \alpha\nabla_b J$$

# Support Vector Machine: Hinge Loss

$$J(\theta) = -\frac{1}{M}\sum_{i=1}^{M}\left[y^{(i)}\log h_\theta(x^{(i)}) + (1-y^{(i)})\log(1-h_\theta(x^{(i)}))\right]$$

$$J(\theta) = \frac{1}{M}\sum_{i=1}^{M}\max(0, 1-y_i(\vec{\theta}\cdot\vec{x}_i+\theta_0))$$

Hinge function

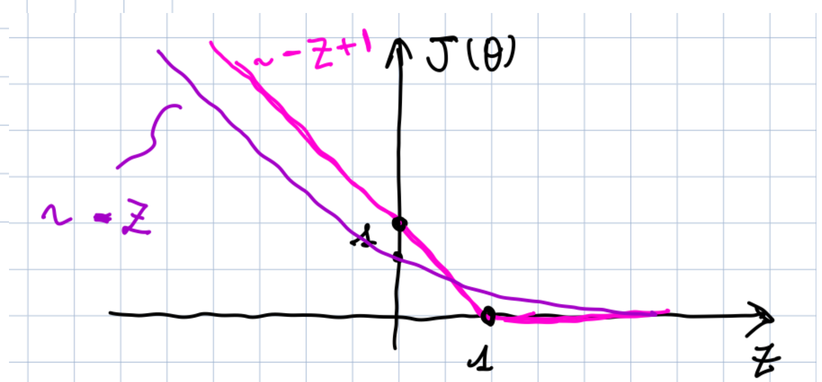
$y_i = 1$ $\quad \vec{\theta}\cdot\vec{x}_i + \theta_0 \geq 1 \quad J(\theta) = 0$

$\quad \vec{\theta}\cdot\vec{x}_i + \theta_0 < 1 \quad J(\theta) = 1-(\vec{\theta}\cdot\vec{x}_i+\theta_0)$

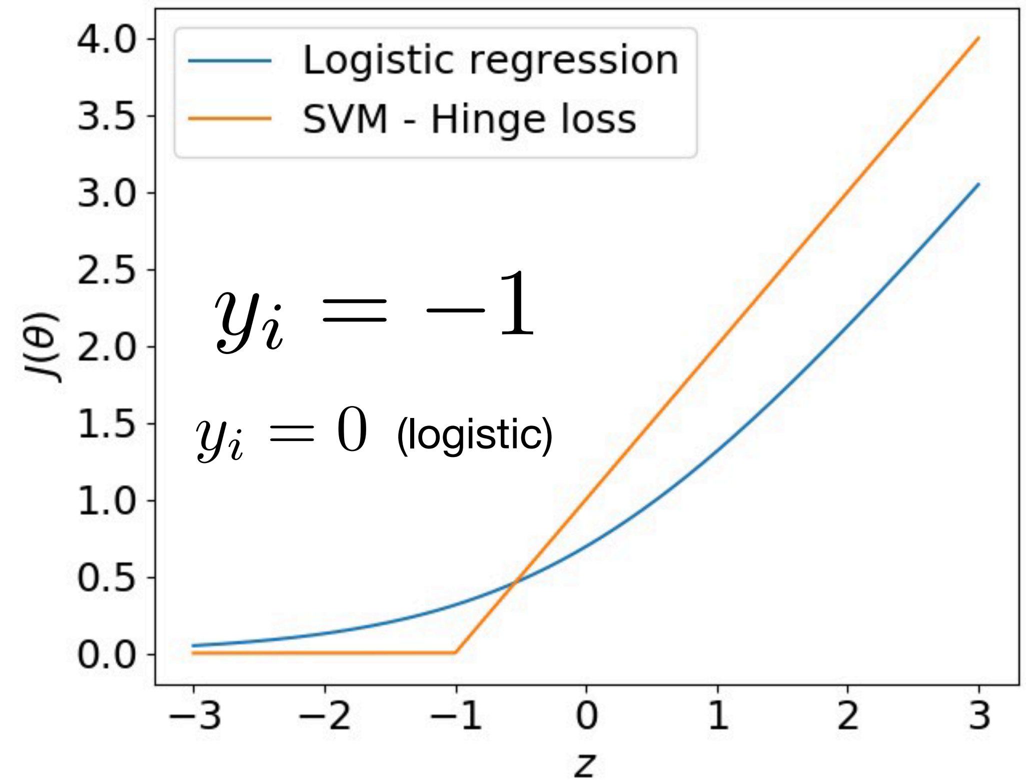
remember in logistic regression

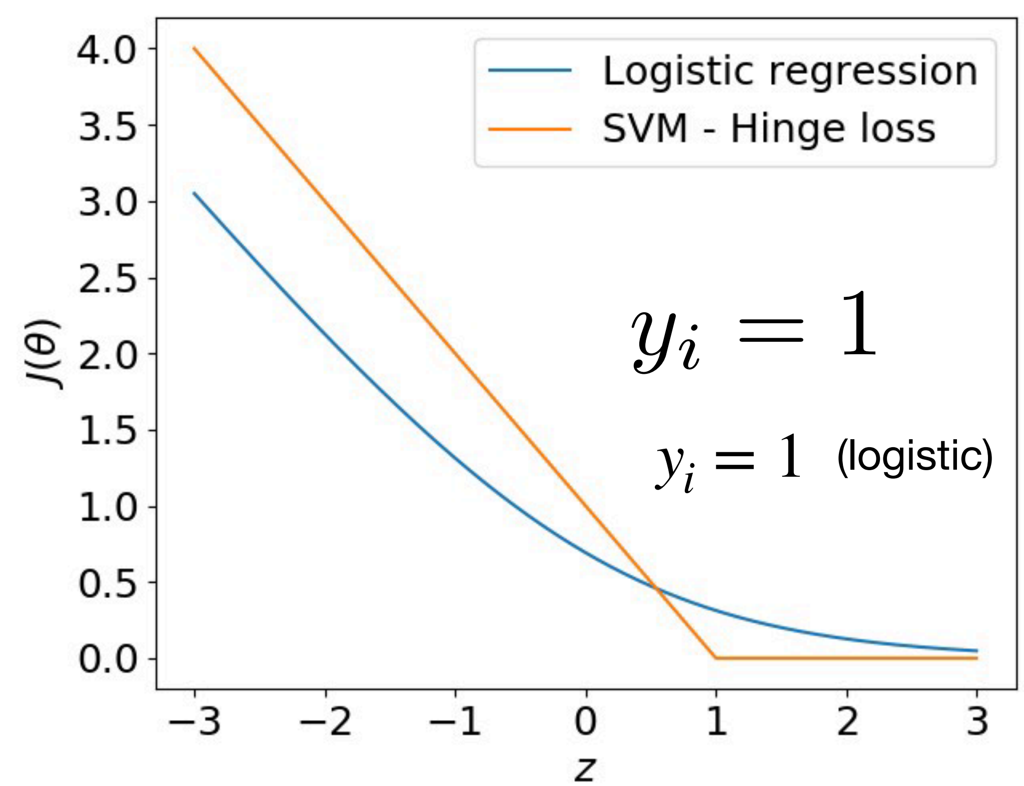
$$J(\theta) = -\log\frac{1}{1+e^{-\theta\cdot x}}$$

$$z = \theta\cdot x = \vec{\theta}\cdot\vec{x} + \theta_0$$



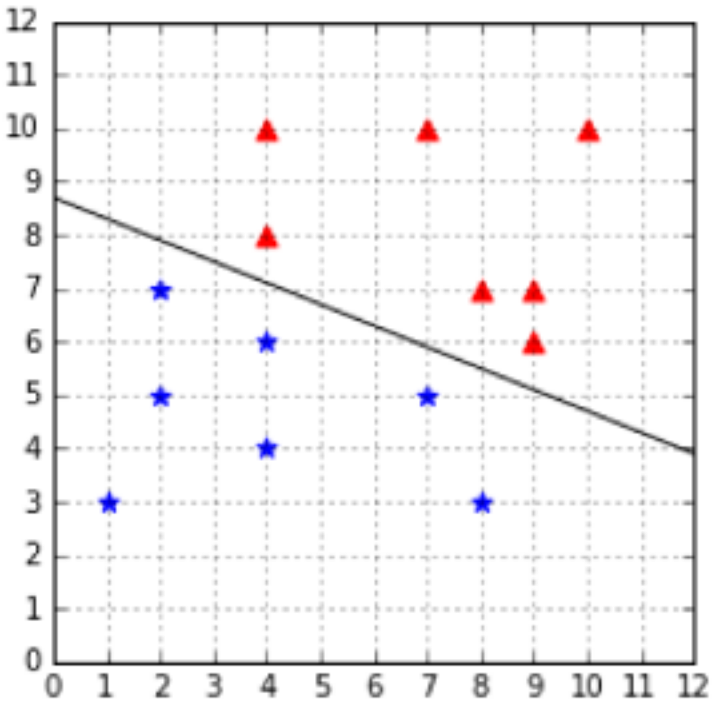




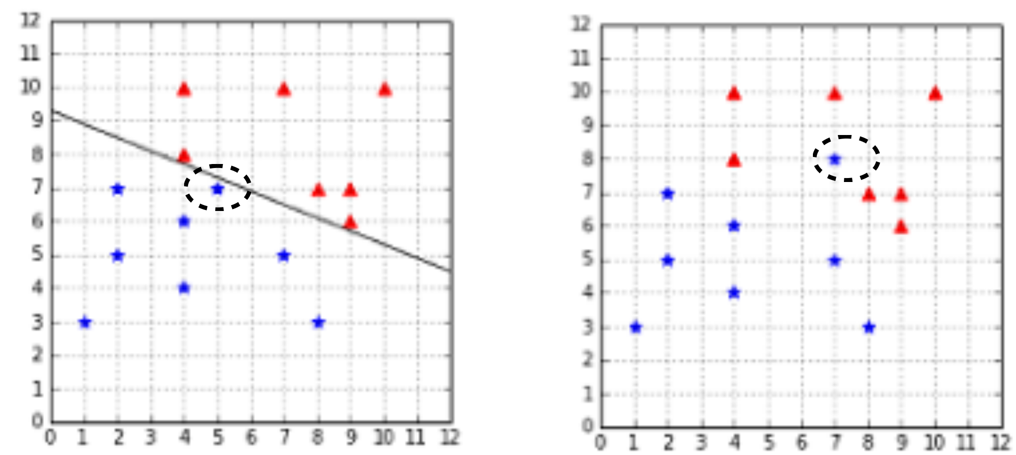



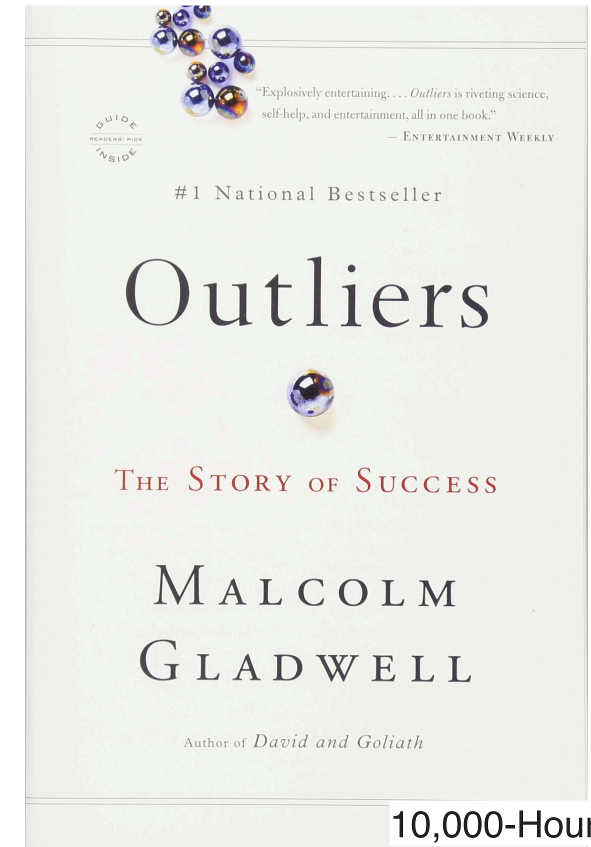
# Support Vector Machine: hard margin and soft margin

## Linearly separable



## Outliers





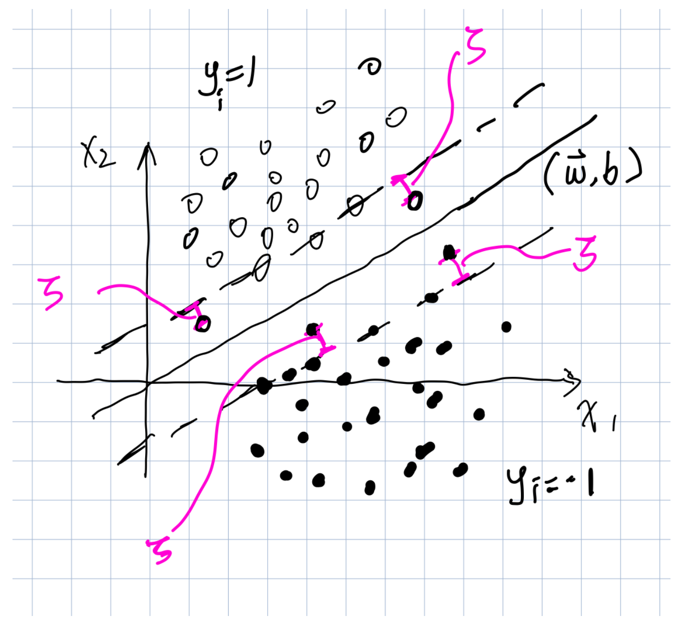

10,000-Hour Rule

## Soft margin to rescue

$$y_i(\vec{\omega} \cdot \vec{x}_i + b) \geq 1$$

$$y_i(\vec{x} \cdot \vec{x}_i + b) \geq 1 - \zeta_i$$

slack variables




Modify the objective function with regularization

$$\underset{(\vec{\omega}, b, \zeta)}{\text{minimize}} \quad \frac{1}{2}||\vec{\omega}||^2 + C\sum_{i=1}^{m} \zeta_i$$

$$\text{subject to} \;\; y_i(\vec{\omega} \cdot \vec{x}_i + b) \geq 1 - \zeta_i$$

$$\zeta_i \geq 0 \;\; \text{for any } i = 1, \cdots, m$$

# Lagrange multipliers and duality

$$\mathcal{L}(\vec{\omega}, b, \alpha, \zeta) = \frac{1}{2}||\vec{\omega}||^2 + C\sum_{i=1}^{m}\zeta_i - \sum_{i=1}^{m}\alpha_i\left[y_i(\vec{\omega}_i \cdot \vec{x}_i + b) - 1 + \zeta_i\right]$$

$$\nabla_{\vec{\omega}}\mathcal{L} = \vec{\omega} - \sum_{i=1}^{m}\alpha_i y_i \vec{x}_i = 0$$

Karush-Kuhn-Tucker (KKT) conditions

$$\frac{\partial \mathcal{L}}{\partial b} = -\sum_{i=1}^{m}\alpha_i y_i = 0$$

$$\frac{\partial \mathcal{L}}{\partial \zeta} = C - \alpha = 0$$

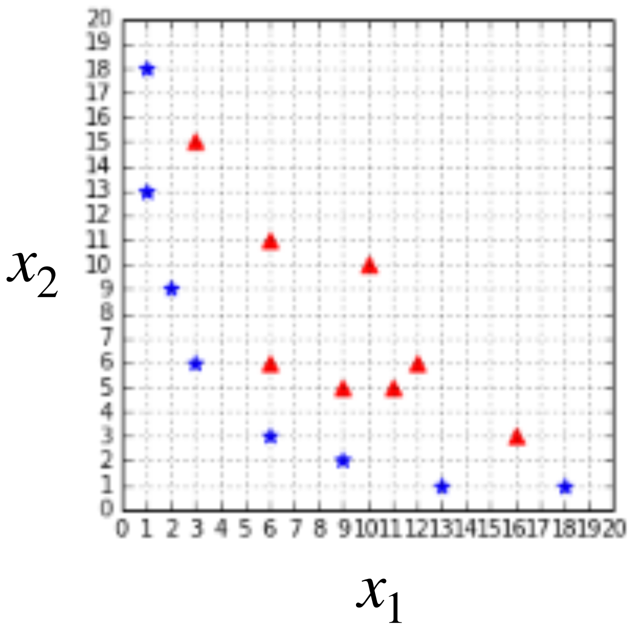
$$\mathcal{L}_D = \sum_{i=1}^{m}\alpha_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j \vec{x}_i \cdot \vec{x}_j$$

$$\max_{\alpha} \mathcal{L}_D(\alpha, \vec{x}_i, y_i)$$

$$\text{subject to} \quad 0 \leq \alpha_i \leq C, \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m}\alpha_i y_i = 0$$

# Kernel machine: Dimensionality reduction strike



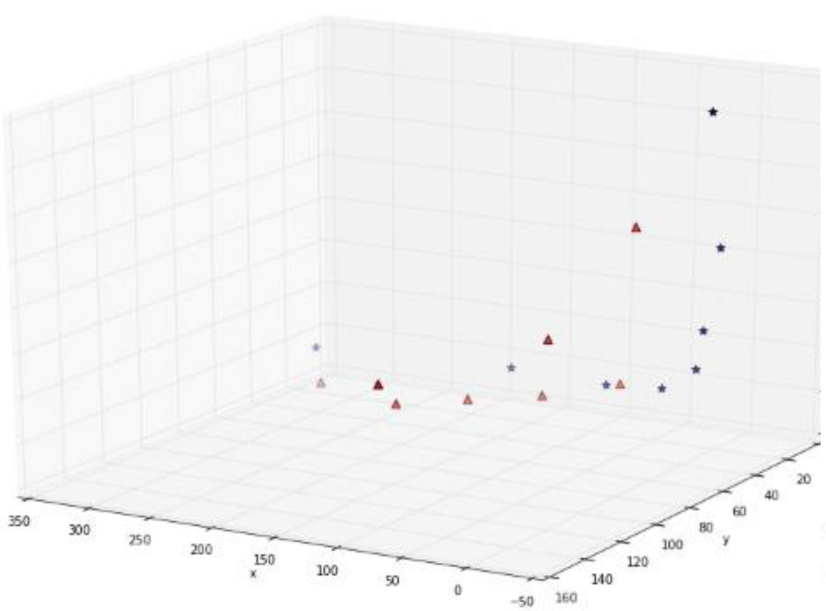

Not linearly separable in **two dimensions**

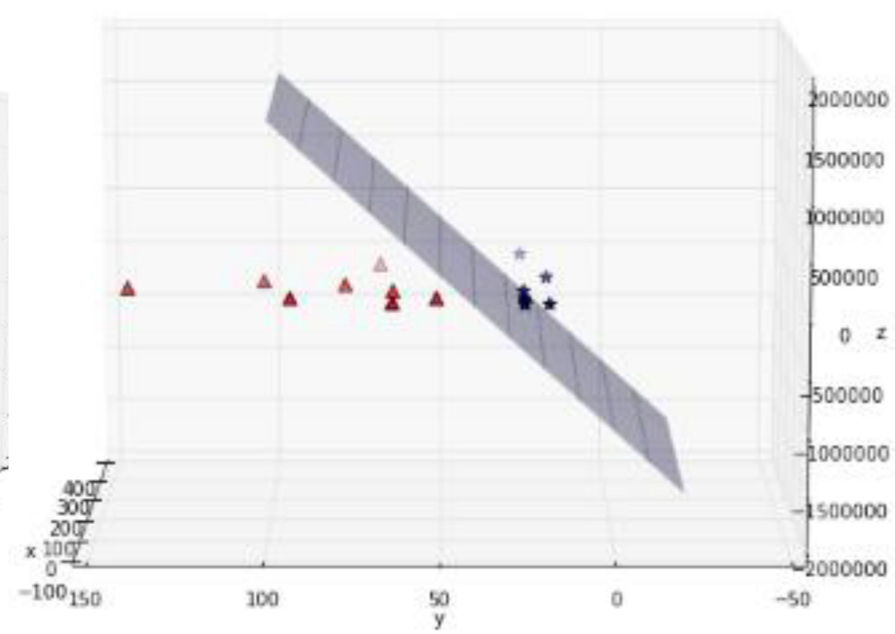
Polynomial mapping $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ $\phi(x_1, x_2) = (x_1^2, \sqrt{2}x_1x_2, x_2^2)$

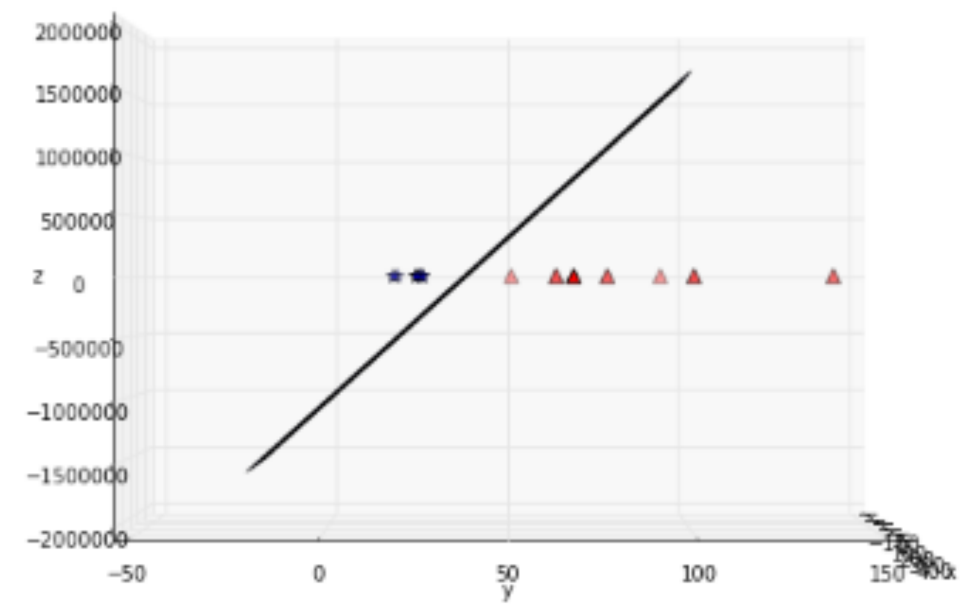
$$\langle \phi(\vec{x}_i), \phi(\vec{x}_j) \rangle_{\mathbb{R}^3} = (x_{i,1}^2, \sqrt{2}x_{i,1}x_{i,2}, x_{i,2}^2) \cdot (x_{j,1}^2, \sqrt{2}x_{j,1}x_{j,2}, x_{j,2}^2)$$

$$= x_{i,1}^2 x_{i,2}^2 + 2x_{i,1}x_{i,2}x_{j,1}x_{j,2} + x_{i,2}^2 x_{j,2}^2$$

$$= (\vec{x}_i \cdot \vec{x}_j + c)^d \quad \text{with } c = 0 \text{ and } d = 2$$







## Kernel machine: Dimensionality reduction strike

mapping $\phi : \mathcal{X} \to \mathcal{V}$

function $K : \mathcal{X} \to \mathbb{R} \quad K(\vec{x}, \vec{x}') = \langle \phi(\vec{x}), \phi(\vec{x}') \rangle_{\mathcal{V}}$

inner production in $\mathcal{V}$, **kernel function**

$$\max_{\alpha} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j K(\vec{x}_i, \vec{x}_j)$$

subject to $0 \le \alpha_i \le C, \ i = 1, \cdots, m$

$$\sum_{i=1}^{m} \alpha_i y_i = 0$$

$$K(\vec{x}, \vec{x}') = (\vec{x} \cdot \vec{x}' + c)^d$$

$c = 1, d = 1$ linear kernel

$c = 0, d = 2$ quadratic kernel