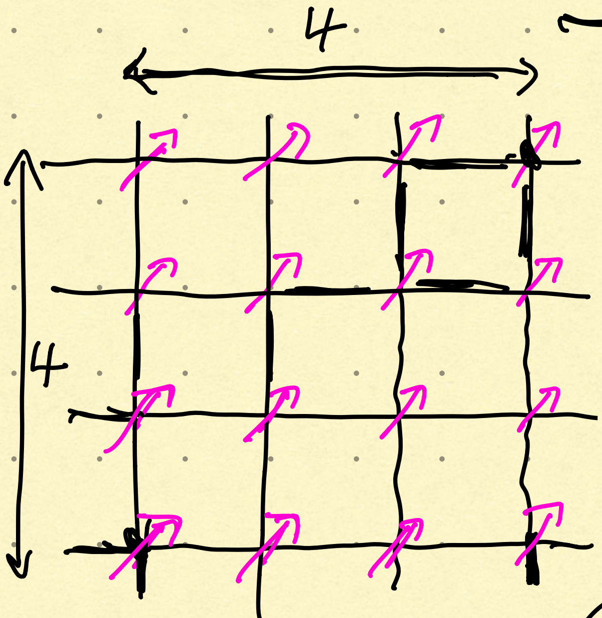
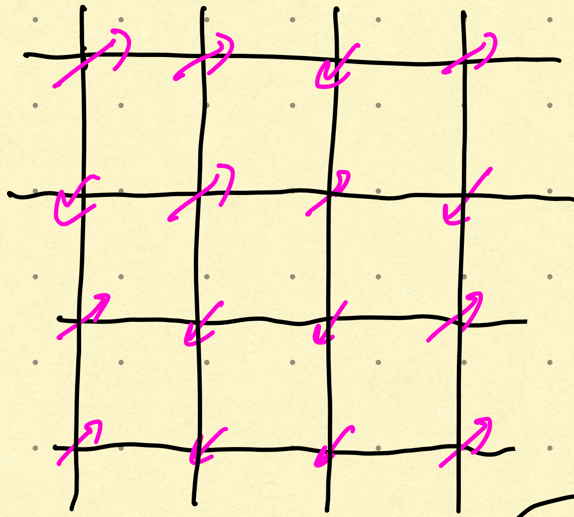
$$H = -J \sum_{\langle i,j \rangle} S_i^z S_j^z \quad , \quad J = 1, \quad S_i^z = \pm 1$$



ferromagnetic order

$T < T_c$



paramagnetic disorder

$T > T_c$

phase transition between disorder and order.

How to understand the transition (continuous)

competition between energy and entropy

$$F = E - TS$$

energy

entropy

for 2D square lattice, in the order phase

at $T = 0$ $\qquad E = -(\#\text{bond}) \cdot (1 \cdot 1) = -2N$

energy is minimized, the entropy is zero.

where $N = L \times L$

In the disordered phase at $T = \infty$

energy is zero due to $+, -$ cancellation,

but entropy can be maximized,

$$S = k_B \ln(\#)$$

$k_B = 1$

$\#$: number of possible state, or, number of configurations

↑ ↓ ↑
↑ ↑
↓ ↑

$$\# = 2^N$$

$$S = N \ln(2)$$

$$\boxed{F = E - TS}$$

$$-2N = -T \cdot N \cdot \ln(2)$$

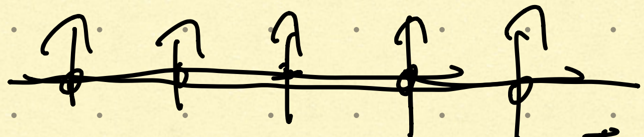
when they are equal

$$2 = T_c \cdot \ln(2)$$

$$T_c = \frac{2}{\ln(2)} = 2.88\cdots$$

Ising

correct value by Onsager (1944)



1D , $T_c = 0$

2D $T_c = \dfrac{2}{\ln(1+\sqrt{2})} = 2.269\ldots$

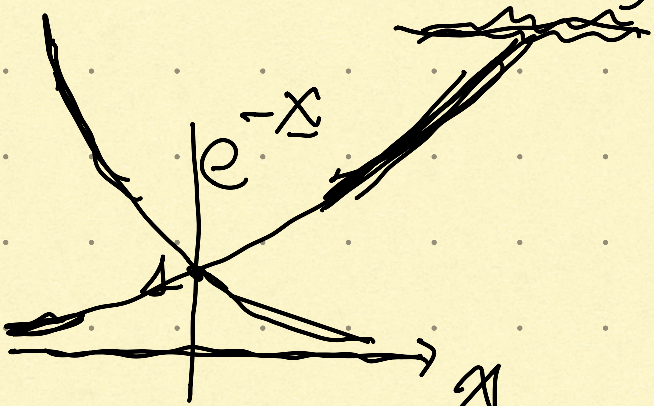
now you obtain this value on your laptop.

Basic Monte Carlo method

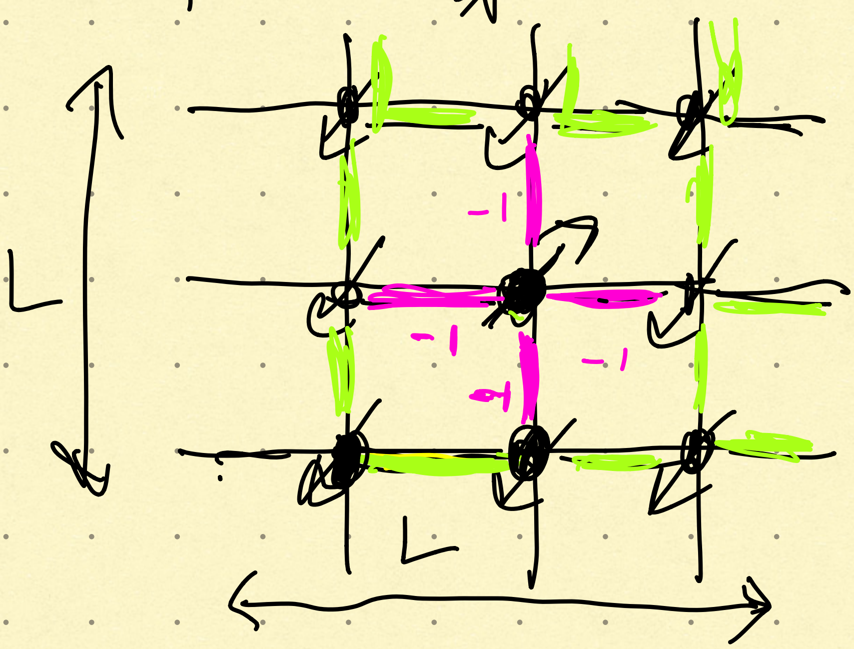
Metropolis - algorithm

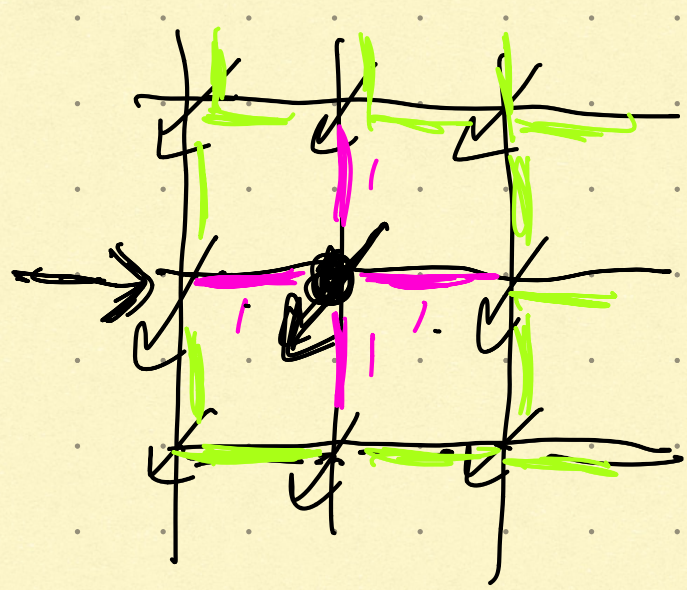
$$E(C) = -\sum_{\langle i,j\rangle \in C} S_i^z S_j^z = \langle H(C)\rangle$$

$$W(C) = e^{-\frac{\overset{1}{\cancel{k_B}}}{T}\boxed{E(C)}}$$



$$= e^{+\frac{1}{T}\boxed{\sum_{\langle i,j\rangle} S_i^z S_j^z}}$$





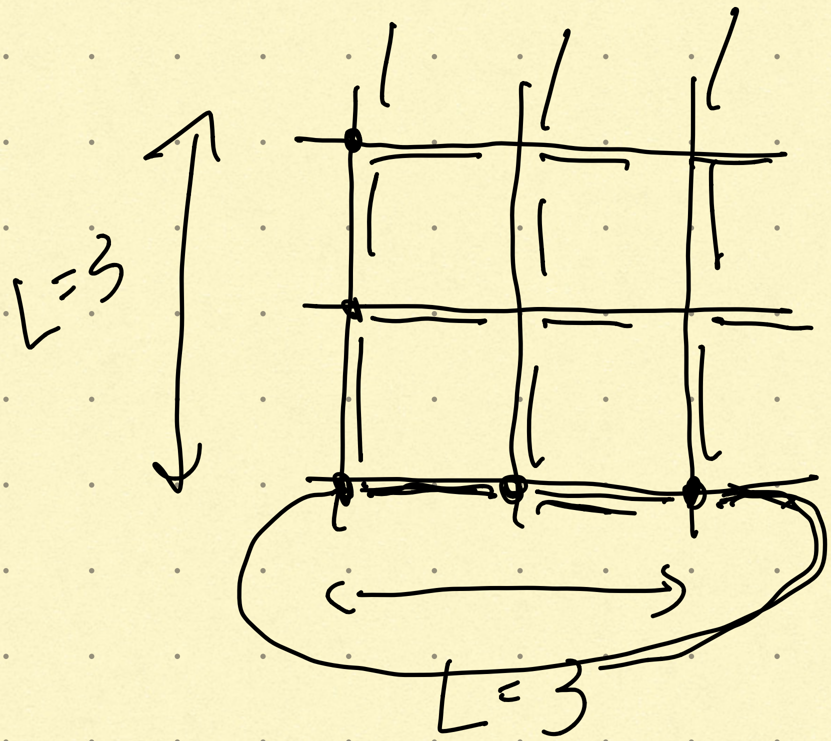
$$C_1 \longrightarrow C_2 \longrightarrow C_3$$

Markov chain

update

$$W(C_1) \;—\; W(C_2)$$

$$= e^{\frac{J}{T}\sum_{\langle ij\rangle} S_i^z S_j^z} \qquad = e^{\frac{J}{T}\sum_{\langle ij\rangle} S_i^z S_j^z}$$



$\#\text{bond} = 2N$

$N = \underset{3}{L} \times \underset{3}{L} = 9$

$2 \cdot 9$

$$\frac{W(C_2)}{W(C_1)} = \frac{e^{\frac{J}{T}\sum_{C_2} S_i^z S_j^z}}{e^{\frac{J}{T}\sum_{C_1} S_i^z S_j^z}}$$

$$= e^{\frac{1}{T}\left[\left(\sum_{C_2} s_i^{\sigma} s_j^{\sigma}\right) - \left(\sum_{C_1} s_i^{\sigma} s_j^{\sigma}\right)\right]}$$

$$4 - (-4) = 8$$

$$= e^{\frac{1}{T}\cdot 8} > 1$$

$$C_1 \to C_2 \to C_3 \to C_4 \to C_5 \to \cdots \to C_{100}$$

$$\frac{W(C_{i+1})}{W(C_i)} = e^{-\frac{1}{T}\left[E(C_{i+1}) - E(C_i)\right]}$$

* $\Delta E < 0$, accept with probability 1

* $\Delta E > 0$, accept with probability

$$0 < e^{-\frac{\Delta E}{T}} < 1$$

C_1 C_2 C_3 ...... 10^6

$|m|(C_1)$ $|m|(C_2)$ ......

$$|m| = \Big|\underbrace{\frac{1}{N}\sum_{i=1}^{N=16}}_{} |s_i^z|\Big|$$

$+1$ -1

$=1$

$=-1$

$$\langle |m| \rangle = \frac{1}{10^6}\sum_{i=1}^{10^6} |m_i| = \mu$$

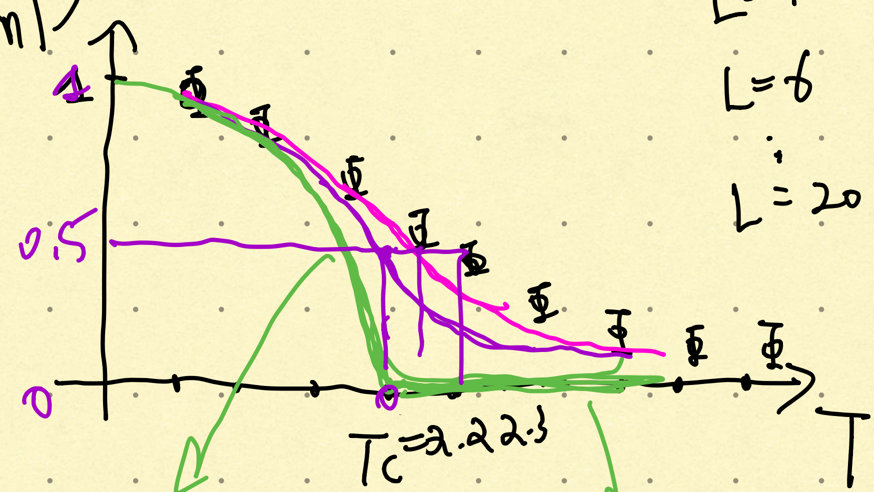
$$\sigma^2 = \sum_{i=1}^{10^6} (|m_i| - \mu)^2 / 10^6$$

$L=4$ $N=4\times4$

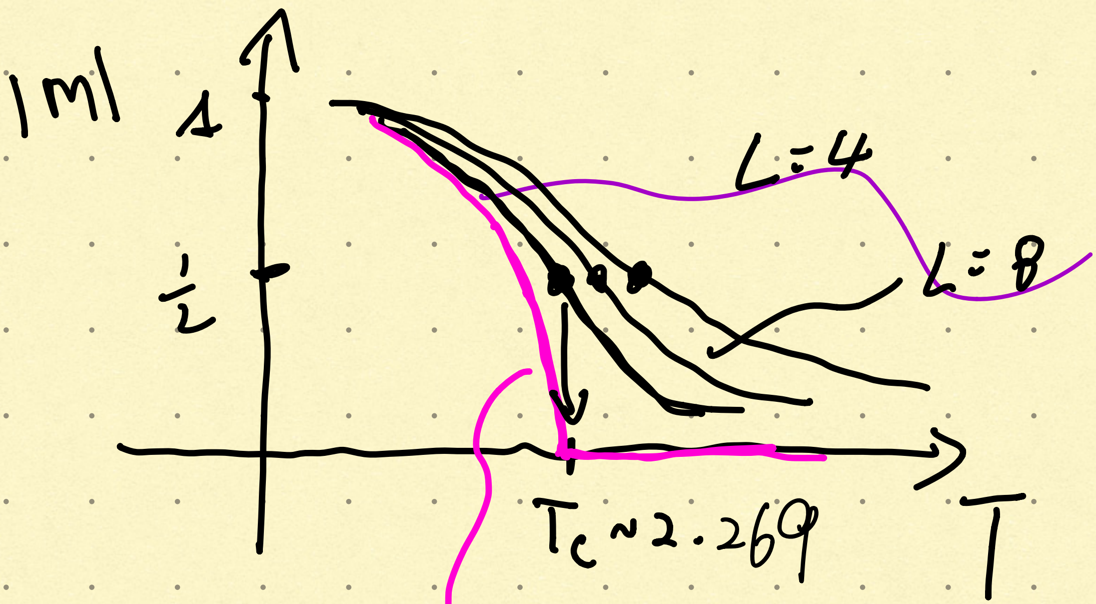
$L=6$ $N=6\times6$

$\vdots$

$L=20$ $N=20\times20$



close to the critical point



$L = \infty$

$$|m| = \boxed{|T - T_c|^{\beta}}$$

T close Tc

β

critical exponent

2.236...

2D Ising

$\beta = \frac{1}{8}$

$\nu = 1$

power-law